CONCRETE STRESS DISTRIBUTION
IN ULTIMATE STRENGTH DESIGN

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and D. McHenry

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CONCRETE STRESS DISTRIBUTION
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Concrete Stress Distribution in Ultimate Strength Design

By EIVIND HOGNETAD, N. W. HANSON, and DOUGLAS McHENRY

SYNOPSIS

Test data are presented which demonstrate the reality and validity of the fundamental plasticity concepts involved in ultimate strength design theories such as those presented by Whitney, Jensen, and others.

A review of earlier experimental investigations regarding the stress distribution in the compression zone of structural concrete flexural members revealed that, though many test methods have been tried, very limited direct test data are available. On the other hand, considerable information has been derived indirectly from strength and behavior observed in tests of reinforced beams and columns.

An eccentrically loaded specimen and a test method were developed which permit the flexural stress distribution to be measured. Complete information regarding the flexural stress distribution, including stress-strain graphs with a descending curve beyond the maximum stress, is reported for w/c ratios of 1.0, 0.67, 0.50, 0.40, and 0.33 at test ages of 7, 14, 28, and 90 days.

INTRODUCTION

The distribution of concrete stress in the compression zone of reinforced concrete members subject to flexure is of fundamental importance in theories regarding the ultimate strength and behavior of such structural members. This stress distribution, often referred to as the stress block, was therefore discussed in the early papers which in about 1900 led to the development of mathematical design procedures for structural concrete. Though the technical details emphasized have changed from time to time, periodic attention has been devoted to the stress block ever since. Nevertheless, it has been possible to gather information only slowly, principally because it is difficult to measure stresses in concrete, although strains may be measured easily and accurately.

The formation in 1952 of the joint ACI-ASCE committee on ultimate strength design was a result of the growing recognition of the significance of the ultimate strength of structural members. To aid the committee in its assignment "to evaluate and correlate theories and data bearing on ultimate
load design procedures with a view to establishing them as accepted practice,” it has again become desirable to improve and extend our knowledge regarding the stress block.

**Historical background**

Several early studies of reinforced concrete beams, such as R. M. von Thullie’s flexural theory of 1897 and W. Ritter’s introduction of the parabolic stress block in 1899, were aimed at developing a theory to explain and permit prediction of ultimate strengths observed in tests. To find a stress-strain relation for concrete, these early investigators turned to concentric compression tests of prismatic plain concrete specimens. They observed deformation and load from zero to the maximum load, and the prism stress-strain relation was then applied to the beam problem.

Such studies and the corresponding emphasis on ultimate strength were discontinued about 1900, at which time the elastic straight-line theory and the concepts of working loads and working stresses became accepted in design throughout the world. Since the modular ratio $n$ is prominently used in the straight-line theory, extensive researches were devoted to the modulus of elasticity of concretes at low loads. Effects of many variables were investigated in concentric compression tests, and various expressions for the modulus of elasticity of concretes and the modular ratio were suggested.

A renewed interest in ultimate strength of structural concrete began about 1930, initiated by F. von Emperger’s critical studies of the modular ratio and working stresses as used in design. Since then, a large number of ultimate strength theories involving a variety of hypothetical stress blocks have been developed. Several theories based on stress-strain relations obtained in concentric prism tests carried beyond the maximum load have also been presented, and some investigators have attempted directly or indirectly to measure the distribution of flexural stresses in tests of reinforced beams. The number of theories suggested between 1930 and 1950 became so large that in 1951 critical reviews were published in both European and American literature.

American studies of ultimate strength of structural concrete have now reached such an advanced stage of development that detailed information regarding the stress distribution in flexure is urgently needed to formulate new design procedures based on ultimate strength and to gain their acceptance in practice. The investigation reported herein was undertaken, therefore, to evaluate previous findings of factual nature and to contribute new data regarding the properties of the stress block.

**Object and scope of investigation**

This investigation was conducted at the Research and Development Laboratories of the Portland Cement Assn. in 1954. The objectives of the study are: (1) to evaluate previous methods and results in experimental investigations of the stress block, and (2) to develop a test method leading to an improved and quantitative understanding of the stress block.
An eccentrically loaded specimen and a test method were developed, and the method was used to measure the properties of the stress block for five concretes with w/c ratios of 1.0, 0.67, 0.50, 0.40, and 0.33 at test ages of 7, 14, 28, and 90 days.

**Notation**

The letter symbols used herein are defined below for convenient reference:

- $A_c$ = concrete gross area
- $A_s$ = area of tension reinforcement
- $A_c'$ = area of compression reinforcement
- $a$ = eccentricity of load
- $b$ = width of rectangular member
- $C$ = total internal compressive force in concrete
- $c$ = distance from neutral axis to compression edge of member
- $d$ = distance from centroid of tension reinforcement to compression edge of member
- $d'$ = distance between centroids of tension and compression reinforcements
- $E_c$ = modulus of elasticity of concrete
- $e'$ = eccentricity of load with respect to centroid of tension reinforcement
- $f_c$ = compressive stress in concrete
- $f_c'$ = compressive strength of 6 x 12-in. cylinders
- $f_s$ = average compressive stress in concrete compression zone
- $f_{cu}$ = stress in compression reinforcement at ultimate load
- $f_y$ = yield point of reinforcement
- $k_1, k_2, k_3$ = coefficients related to magnitude and position of internal compressive force in concrete compression zone (Fig. 1)
- $k_a = e/d$ ratio indicating position of neutral axis at failure
- $M$ = bending moment
- $M_{ult}$ = ultimate bending moment
- $m_a = M/bc^2$, modified moment term
- $P$ = Load
- $P_{ult}$ = ultimate load
- $p = A_p/bd$, reinforcement ratio
- $q = A_p f_y/bd f_c'$, tension reinforcement index
- $q' = A_p f_y/bd f_c$, compression reinforcement index
- $\%_c$ = strain in concrete
- $\%_{cu}$ = strain in reinforcement at ultimate load
- $\%_{ul}$ = strain in reinforcement at ultimate load
- $\%_p$ = steel strain at initial yielding

**FLEXURAL STRENGTH OF STRUCTURAL CONCRETE**

**Basic ultimate strength equations**

To illustrate the properties of the stress block that are most important for practical purposes, the basic equations for the flexural strength of structural concrete are reviewed. An analytical approach originated by F. Stüssi is used, which through refinements contributed later by others has been made applicable to flexure of reinforced concrete with and without axial load as well as to prestressed concrete.

Stress conditions at the ultimate load capacity of a rectangular structural concrete member subject to combined flexural and axial load are shown in Fig. 1. Equilibrium of moments and forces is expressed by

\[
M_{ult} = P_{ult} e' = k_1 k_2 f_c' (d - e_p) + A_s f_{cu} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
If failure is initiated by yielding of the tension reinforcement, the steel stress at the ultimate moment $f_{tu}$ equals the yield point $f_y$, and with no compression reinforcement we obtain by solving Eq. (1) and (2)

$$\frac{M_{ult}}{bd f'_t} = q \left(1 - \frac{k_2}{k_1 k_3} q\right)$$

(3)

in which the tension reinforcement index

$$q = \frac{A_t f_y}{bd f'_t} = \frac{f_{tu}}{f'_t}$$

Similarly, if both tension and compression reinforcement of an eccentrically loaded member are yielding at failure, we obtain

$$\frac{P_{ult}}{bd f'_t} = q' - q + \frac{k_1 k_2}{2k_3} \left[ -\left(\frac{d'}{d} - 1\right) + \sqrt{\left(\frac{d'}{d} - 1\right)^2 + \frac{4k_2}{k_1 k_3} \left(q' + \frac{d'}{d} - 1\right)} \right]$$

(4)

in which the indexes are

$$q = \frac{A_1 f_y}{bd f'_t} \text{ and } q' = \frac{A'_1 f'_t}{bd f'_t}$$

In the equations for tension failure, Eq. (3) and (4), the only property of the stress block needed is the ratio $k_2/k_1 k_3$.

For compression failures, crushing of the concrete takes place before yielding of the tension reinforcement. It is then necessary to consider strains to determine the stress in the tension reinforcement at failure. Assuming a linear distribution of strain, we obtain with the notation of Fig. 1

$$k_n = \frac{c - e_n}{d - \epsilon_u + \epsilon_u}$$

(5)

Combining Eq. (5) with Eq. (1) and (2), and assuming that the compression reinforcement is yielding at failure, the tensile stress $f_{tu}$ as well as $P_{ult}$ and $M_{ult}$ can be determined. It is then necessary to know $\epsilon_u$ and independent values of $k_4$ and $k_1 k_3$, not only the ratio $k_2/k_1 k_3$.

If it is assumed that the concrete carries no tension below the neutral axis, Eq. (1) to (5) apply also to T-sections and to hollow sections, provided that the neutral axis at ultimate load is not below the compression flange. For prestressed concrete, an equation similar to Eq. (5) can be used in conjunction with the stress-strain relation for the prestressing steel to determine the steel stress at failure.\(^*\)

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\(^*\) Fig. 1—Conditions at ultimate load
Therefore, an analysis of this type is a powerful design tool provided that the properties of the stress block are known at least to the extent of $k_2$, $k_1k_3$, and $\epsilon_u$. If the entire shape of the stress block is known, any shape of section and nonsymmetrical bending of rectangular sections can be analyzed.

**Experimental studies of stress block**

It is difficult to measure stresses in concrete at high loads. Strains can of course be measured easily by various types of gages, but stresses at high loads are not proportional to strains. Many methods have been tried, therefore, to obtain experimental data regarding the distribution of concrete stresses in flexure.

*Concentric compression tests*—A great number of mathematical expressions have been developed for the stress-strain relationship for concentrically loaded prisms. Most of these expressions consider the range from zero to the maximum stress only, since final collapse of compression test specimens often takes place shortly after the maximum stress is reached. Prism stress-strain relations have been applied to bending of reinforced concrete by assuming a linear distribution of strain in the compression zone. Some authors have assumed that the extreme "fibers" in a beam at the ultimate moment are subject to a maximum stress and a corresponding strain which are both similar to those determined in a simple compression test. Others have recognized, however, that an ultimate strain can be developed which is greater than the strain at the maximum stress in a conventional compression test. It has been shown that the sudden failures observed in compression tests are related to the release of energy stored in the testing machine, which in turn is related to the stiffness of the machine. By using suitably stiff testing machines, or by surrounding the concrete specimen by a system of steel springs, stress-strain relations have been observed beyond the maximum load. An example of such a relation is given in Fig. 2.

![Fig. 2—Tests of 3 x 6-in. concrete cylinders—U. S. Bureau of Reclamation](image-url)
Application of stress-strain relations for concentric compression to flexure of reinforced concrete has been vigorously challenged. A plain concrete specimen which has been strained beyond the maximum load in concentric compression is generally cracked, and the strain response to load is highly sensitive to time. It has been asserted, therefore, that concrete strained beyond the maximum point of the stress-strain curve is useless for load-carrying purposes in beams as well as in columns. An opposite view has been expressed by some who point out that special testing techniques are required to determine the true stress-strain relationship in concentric compression, but that the general characteristics as shown by Fig. 2 are applicable to flexure of reinforced concrete.

Bending-simulation machine—A special testing machine has been constructed at Imperial College, London, to study the behavior of reinforced concrete members. The action of compressive stresses in the concrete compression zone of a prismatic specimen are simulated by six pairs of hydraulic jacks operated by six independent pump systems.

Tests are carried out by increasing the load in stages, and at each stage finding by trial the distribution of jack loads corresponding to a linear distribution of strains in the specimen as indicated by strain gages. In this manner the machine has been used to study the shape of the stress block. Only limited results have been published; research with the machine is being continued.

Photoelastic methods—The distribution of concrete stresses has also been studied by embedding a piece of special glass in the compression zone of reinforced concrete beams. Photoelastic patterns obtained by passing polarized light through the glass plate at various beam loads gave information from which the distribution of compressive stresses was determined. It was found, however, that variations in moisture and humidity conditions greatly affect the observed stress distributions since volume changes of the concrete influenced stresses in the glass plate. The experimental difficulties involved in this method are therefore considerable, and the data obtained must be interpreted cautiously.

Stress meters—An attempt has been made to measure stresses by embedding a stack of eight steel plates in the compression zone of a reinforced concrete beam. At both ends the plates had the same width as the beam, while the central section where strain gages were located was reduced to give a “modulus” similar to that of the concrete. Interpreting the strain readings in the individual steel plates in terms of concrete stresses, a straight-line stress distribution was found even at high loads. The principal weakness of this experimental method is the unknown extent to which the introduction of the steel plates locally changes the stress conditions in the concrete.

A more satisfactory method was developed by the U. S. Bureau of Reclamation. Small pressure cells were embedded in reinforced concrete beams. Test results indicate that although strain is linearly distributed even near failure, the distribution of compressive stress is curvilinear with the maxi-
mum measured stress equal to the cylinder strength. The use of such pressure cells is a promising approach to studies of the stress block. The need for fairly complex and expensive instrumentation is probably the principal reason that the use of this approach has been limited.

Tests of reinforced concrete—Most of the available data regarding properties of the stress block have been obtained from tests of reinforced concrete members. Strains have been measured directly with the aid of strain gages. Information regarding stress distribution, however, has generally been derived indirectly by computing from the requirements of equilibrium those properties of the stress block which were consistent with the observed behavior and strength of reinforced members.

A large number of tests have indicated that the distribution of strain in structural concrete is essentially linear even near ultimate load. In this respect most investigators have agreed. On the other hand, expressions that have been suggested for the ultimate strain \( \epsilon_u \) at the compression face vary considerably. For instance, Saliger\(^{19} \) indicated that \( \epsilon_u \) is proportional to compressive strength, while Brandtzaeg,\(^{29} \) Ros,\(^{21} \) and Jensen\(^{22} \) indicated a decrease of \( \epsilon_u \) with increasing strength. In recent American tests\(^{8,9,23} \) it has been found that \( \epsilon_u \) is largely independent of compressive strength, and average \( \epsilon_u \) values of 0.0034 to 0.0038 have been suggested.

The ratio \( k_2/k_1k_3 \) has been studied extensively since this ratio can be determined by Eq. (3) when \( M_{ut}, f', f_c \), and beam dimensions are known. Values from 0.5 to 0.6 have usually been suggested.\(^{1,2} \)

The value \( k_1k_3 \) has been evaluated from beam test data by measuring the depth \( c \) to the neutral axis at the ultimate moment and assuming a value of \( k_2 \). Then

\[
T = C = \frac{M_{ut}}{d - k_3} = k_1k_3f' \text{bc} \quad \text{...................................................(1a)}
\]

Since \( M_{ut} \) is measured, \( k_1k_3 \) can be evaluated. In this manner Billet and Appleton\(^{8} \) found

\[
k_1k_3 = \frac{3000 + 0.5 f'}{1500 + f'} \quad \text{...................................................(6)}
\]

and for \( f' \) greater than 2000 psi Gaston\(^{23} \) obtained

\[
k_1k_3 = \frac{600}{f' - 1500} \quad \text{...................................................(7)}
\]

Prentis, Hamann, and Leo\(^{24-26} \) have derived further information regarding the stress block from beam test data by numerical differentiation. The methods used by these three investigators are related, and Lee's derivation typifies the principles involved. Four major assumptions were made:

1. The reinforcement is elastic to the yield point, \( f_y = E \epsilon_y \). Beyond the yield strain \( \epsilon_y \), stress is constant and equal to \( f_y \).
2. Concrete does not resist tension.
3. Linear distribution of strain.
4. Concrete compressive stress is a function of strain only, \( f = F(\epsilon) \). Effects of time and of strain gradient are neglected.
With the notation of Fig. 3, the position of the neutral axis is then given by

$$\frac{c}{d} = \frac{c_0}{c} + \epsilon_0$$

Internal equilibrium of forces gives

$$C = b \int_0^c F(\epsilon_s) \, dx - \frac{bc}{\epsilon_s} \int_0^{\epsilon_s} F(\epsilon_s) \, d\epsilon_s = T = \begin{cases} A_E \epsilon_s & \text{for } 0 < \epsilon_s < \epsilon_y \\ A_f \delta & \text{for } \epsilon_s > \epsilon_y \end{cases}$$

Substituting Eq. (8) into Eq. (9), differentiating with respect to $\epsilon_s$ and rearranging

$$f_s = \frac{A_s}{bd} E_s \left( \epsilon_s \frac{d\epsilon_s}{d\epsilon_s} + 2 \epsilon_s \frac{d\epsilon_s}{d\epsilon_s} + \epsilon_s \right)$$

$$f_s = \frac{A_s}{bd} f_s \left( 1 + \frac{d\epsilon_s}{d\epsilon_s} \right)$$

If $\epsilon_s$ and $\epsilon_0$ are observed in test beams loaded in small increments, a relation between stress $f_s$ and strain $\epsilon_s$ may be established by Eq. (10) and (11) when the differentials $d\epsilon_s/d\epsilon_s$ are replaced by finite differences $\Delta\epsilon_s/\Delta\epsilon_s$.

Each of the three investigators presented only a single application of the mathematical methods to test data, and promising results were obtained. Certain experimental errors, however, tend to be strongly amplified in numerical differentiation of this type. For instance, measured values of the steel strain $\epsilon_s$ are influenced by the position of strain gages relative to cracks in the concrete tension zone, and some tensile stresses are carried by the concrete even at high loads. At low loads the concrete is not cracked and Eq. (10) and (11) do not apply.

It was felt that these deficiencies could be overcome by modifying the testing procedures, particularly by avoiding the complications introduced by the presence of reinforcement and tensile stresses in the concrete.

**METHODS OF TESTING AND ANALYSIS**

Prentis applied numerical differentiation to test data from an unbonded prestressed beam, which permits more accurate steel strain measurements than are possible for embedded bar reinforcement. The same principle was used in the present investigation by applying a major thrust $P_1$ by a testing machine (Fig. 4). By further applying a minor thrust $P_2$ that could be varied independently of $P_1$, the neutral axis could be maintained at a face of the test specimen throughout the test. Thereby complications resulting from tensile stresses in the concrete were eliminated.
Considering the stiffness characteristics of the testing machine used to apply the major thrust \( P_1 \), the relative positions of \( P_1 \) and \( P_2 \) as well as the stiffness of the tie rod by which \( P_2 \) was applied were so chosen that no uncontrolled release of strain energy could take place. These considerations led to a specimen with dimensions as given in Fig. 4. The central un reinforced test region has a cross section 5 x 8 in., and is 16 in. long. The sole purpose of the reinforced brackets at the ends of the specimen is to accommodate the two thrusts.

**Method of testing**

Suitable tension, compression, and shear reinforcement was placed in the end brackets to obtain failure in the central un reinforced test region. All such reinforcement ended at least 4 in. from the test region. After curing, dimensions of the test region were measured carefully by vernier calipers, and actual areas rather than the nominal area of 40 sq in. were used in all calculations.

Since numerical differentiation, which tends to amplify experimental scatter, was to be applied to the test data, all testing operations were carried out with unusual care. As shown in Fig. 5, the major thrust \( P_1 \) was applied by a testing machine. The specimens were carefully leveled in the machine, and the load was applied through a 3/16-in. steel roller. The minor thrust \( P_2 \) was introduced by a hydraulic jack through one or two tie rods, and the load was measured by a calibrated electric strain gage transducer.

Strains were measured by two 6-in. electric strain gages at the neutral surface, two gages at the compression surface, and one gage at mid-depth of each of the two side faces (Fig. 4). All of these strains as well as the output of the tie rod transducer were recorded continuously throughout the test period by an electronic recorder. Transverse strain on the compression face was also recorded. Finally, the deflection at mid-height was measured by a dial gage so that the eccentricities of the thrusts were known at all load levels.

The main thrust \( P_1 \) was applied at a steady rate from zero to failure during a testing period of about 15 min. By operating a hydraulic pump, the tie rod force was adjusted continuously so that the strain at the neutral face was maintained at zero within 5 millionths.
Three or four 6 x 12-in. cylinders were tested with each specimen. Two 6-in. electric strain gages were mounted on one cylinder of each group, and strains were recorded as a function of load in a testing machine with known stiffness characteristics.

Analysis

Without assumptions of any kind we obtain by equilibrium of forces and moments (Fig. 4)

\[
k_1k_2 = \frac{C}{bc f_c'} = \frac{P_1 + P_2}{bc f_c'} \tag{12}
\]

\[
k_2 = 1 - \frac{P_1a_1 + P_2a_2}{(P_1 + P_2)c} \tag{13}
\]

in which the eccentricities \(a_1\) and \(a_2\) include both the initial values and those due to deflection of the specimen. In this manner the important properties of the stress block \(k_1k_2\) and \(k_2/k_1k_2\) may be measured directly from zero load to failure.

The complete stress-strain curves in flexure may be obtained if two reasonable assumptions are made:

1. Linear distribution of strain across the test section. This has been well verified by numerous earlier tests and was confirmed also by the tests reported herein.
2. All "fibers" follow one and the same stress-strain curve. In other words, concrete stress is a function of strain only, \( f = F(\varepsilon) \).

We then obtain by equilibrium of forces and of moments (Fig. 3 and 4)

\[
C = b \int_0^{\varepsilon_c} F(\varepsilon_x) \, d\varepsilon_x - \epsilon_c^2 \int_0^{\varepsilon_c} F(\varepsilon_x) \, d\varepsilon_x = P_1 + P_3 = f_c b c \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots 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### TABLE 1—CONCRETE MIXTURES

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<tr>
<th>Concrete</th>
<th>Age at test, days</th>
<th>Cylinder strength, ksi psi</th>
<th>Slump, in.</th>
<th>Percent sand by weight of total aggregate</th>
<th>Cement, sacks per cu yd</th>
<th>Water, lb per cu yd</th>
<th>w/c by weight</th>
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**Materials**

The cement used in making the concretes was a blend of Type I cements. Fine aggregate was Elgin sand modified by a small amount of lake sand to improve the grading. Coarse aggregate was a gravel of 1½-in. maximum size. The gravel was separated into three fractions and recombined. The aggregates are a partly siliceous and partly calcareous material of glacial origin from Elgin, Ill.

The concretes were mixed in either a 2-cu ft tilting drum mixer or a 6-cu ft nontilting mixer. Mix proportions are given in Table 1. Specimens were cast in a horizontal position. The concretes were placed in plastic-coated forms and in the 6 x 12-in. cylinder molds by an internal spud vibrator. The surface was screeded about 15 min after casting, and was later lightly troweled. After about 18 hr, the specimens and cylinders were covered by moist burlap for initial curing. Forms were removed two days after casting, and the specimens and their companion cylinders were placed in a moist room for the remainder of the curing period. Five days before testing, drying was started by storage in the laboratory at a temperature of about 75 F and a relative humidity of 50 to 80 percent, and all specimens were tested dry. This period of drying was necessitated by instrumentation requirements.

**TEST RESULTS**

Test data recorded for each specimen gave strain in the prismatic test region and the tie rod force $P_2$ as a function of the major thrust $P_1$ (Fig. 4). The eccentricities $a_1$ and $a_2$ were so chosen that $P_1$ always increased until a maximum value was reached at failure. To maintain the neutral axis at a face of the specimen, $P_1$ was initially increased with $P_1$ until a maximum was
reached, after which $P_2$ decreased again. The eccentricity $\alpha_2$ and stiffness of the tie rods were always so chosen that $P_2$ tended to decrease rapidly as deflection of the specimen shortened the tie rods. Thus, though $P_2$ decreased near failure, the piston of the hydraulic jack in the tie rod system was continuously extended throughout every test. In this manner no uncontrolled release of strain energy took place. There are strong reasons to believe, therefore, that the observed phenomena are related only to properties of the concrete test specimen, not to the elastic properties of the loading equipment and the testing machine.

Fig. 6 shows the values of $k_1k_3$ and $k_2$, computed by Eq. (12) and (13), as a function of the strain $\varepsilon_c$ at the compression face. These curves are given only for the 90-day specimens, but the nature of the curves obtained at other test ages was similar. The smooth and steady increase of the $k_1k_3$ curves in Fig. 6 indicates a rise of the average concrete stress to about 90 percent of the cylinder strength for a high water-cement ratio and to about 65 percent for low w/c ratios. These values represent the average for the entire depth from the neutral surface to the extreme fiber. At low loads $k_3$ is near the value $1/3$ corresponding to a triangular stress distribution. With increasing load, however, the position of the resultant internal compressive force changes, and
Fig. 7—Typical strain profiles

for the high \( w/c \) ratios \( k_2 \) reaches the value 1/2, which value could also be obtained by a rectangular stress distribution.

Strain distributions across the test section of four typical specimens are given in Fig. 7. A linear distribution of strains was found even at high loads, as has been shown by numerous previous tests of reinforced concrete beams and columns.

**Ultimate strength design values**

Three stress block properties of fundamental importance in ultimate strength design—\( k_3/k_2 \), \( k_2/k_1k_3 \), and \( \varepsilon_u \)—are given in Fig. 8 as a function of cylinder strength and with test age as a parameter. All quantities correspond to the maximum values of \( k_3/k_2 \) obtained in each test, since a slight decrease in \( k_3/k_2 \) and an increase in \( k_2 \) at times took place near failure while visible crushing was in progress. Values of \( k_3/k_2 \) derived from other tests of reinforced concrete as well as values of \( \varepsilon_u \) measured in such tests are also given in the figure, and various curves representing equations given by previous investigators are shown.

\( k_3/k_2 \) values—The 20 points of this investigation shown in Fig. 8a reveal that \( k_3/k_2 \) is a function of \( f'_c \) regardless of concrete age. These points form a rather narrow band in the lower scatter fringe of the points derived by Billet and Appleton\(^2\) and Gaston\(^2\) from reinforced concrete beam test data. A decrease of \( k_3/k_2 \) with increasing cylinder strength is evident, a variation which may be expressed by

\[
    k_3/k_2 = \frac{3900 + 0.35 f'_c}{3200 + f'_c} \quad \text{--------} \quad (18)
\]

Eq. (18) gives lower \( k_3/k_2 \) values than those obtained by Billet and Appleton's Eq. (6) and Gaston's Eq. (7), which were derived indirectly from beam tests.
Fig. 8—Ultimate strength properties of stress block
However, a close agreement exists between Eq. (18) and the $k_1k_3$ values obtained on a theoretical basis by Jensen. By assuming a trapezoidal stress block with a maximum stress equal to the cylinder strength ($k_3 = 1.0$), Jensen obtained

$$k_1k_3 = \frac{1 + \beta}{2}$$

(19)

in which the plasticity ratio $\beta$ was derived from measured ultimate moments of reinforced concrete beams as

$$\beta = \frac{1}{1 + \left(\frac{f_c}{4000}\right)^2}$$

(20)

A curve representing $k_1k_3$ values derived by Hognestad from tests of eccentrically loaded reinforced columns is also shown in Fig. 8a. For low concrete strengths, he found low values of $k_1k_3$ as compared to Eq. (18). It is believed that the principal reason for this difference is that the prisms of the present investigation were cast in a horizontal position whereas the columns of Hognestad's investigation were 6 to 7 ft high and were cast in a vertical position. Concretes with a slump of about 6 in. were used in the latter tests, and water gain probably led to a reduction in strength in the upper part of the columns. All failures occurred in the upper half, and concentrations of curvature and strains were found near the tops of the columns. A similar bleeding effect was observed by Kennedy in vertically cast cylinders 32 in. high which were sawed into four 8-in. cylinders for testing. Compressive strength of the top cylinder was about 25 percent lower than that of the bottom cylinder.

$k_0/k_1k_3$ values—The ultimate moment of beams failing in tension is relatively insensitive to variations of $k_2/k_1k_3$. If this ratio is computed from individual measured ultimate moments by Eq. (3), therefore, experimental scatter will be amplified strongly. In investigations of the ultimate strength of beams, $k_2/k_1k_3$ has usually been derived by studying test results of large groups of beam test data, assuming that $k_2/k_1k_3$ is independent of concrete strength. For instance, Whitney found $k_2/k_1k_3 = 0.5/0.85 = 0.59$ (Fig. 8b). On the basis of column tests Hognestad found $k_2/k_1k_3 = 0.65$. Jensen found an increase of $k_2/k_1k_3$ with cylinder strength as shown in Fig. 8b; however, he considered it sufficiently accurate for practical purposes to take $k_2/k_1k_3 = 1/2$. The value of 1/2 has been much used in foreign literature.

Data obtained in the present investigation (Fig. 8b) indicate $k_2/k_1k_3$ is a function of concrete strength regardless of age:

$$\frac{k_2}{k_1k_3} = \frac{1600 + 0.46 f_c'/80,000}{3900 + 0.35 f_c'}$$

(21)

For practical design purposes Whitney's choice of constant value $k_2/k_1k_3 = 0.59$, which gives satisfactory agreement with beam tests, should suffice.

Ultimate strains—The ultimate strains $\varepsilon_u$ measured in this investigation are compared in Fig. 8c to those of numerous earlier tests of reinforced con-
crete columns and beams. Again, our points form a rather narrow band in
the lower scatter fringe of the reinforced concrete test data, and no systematic
variation of \( \varepsilon_u \) with concrete age is found. A variation of \( \varepsilon_u \) with concrete
strength is evident, which may be expressed by
\[
\varepsilon_u = 0.004 - f_u/6.5 \times 10^4
\] (22)
For design purposes a constant value \( \varepsilon_u = 0.003 \) is believed to be sufficiently
conservative since the computed ultimate strength of reinforced concrete
members is usually relatively insensitive to the numerical value of \( \varepsilon_u \).4

Balanced steel ratio in beams—In practice, relatively low reinforcement ratios
are generally used in reinforced concrete beams. Failure is then initiated by
yielding of the tension reinforcement, and ultimate strength design may be
carried out by Eq. (3).

The applicability of Eq. (3) ceases when sufficient reinforcement is present
to cause a compression failure. To evaluate the balanced steel ratio, \( p_b \),
we have for rectangular beams

\[
k_u = \frac{\varepsilon_u}{\varepsilon_t + \varepsilon_u}
\] (5a)

and

\[
C = k_b f_u k_u b d = T = A_s f_y
\] (1b)
which gives

\[
p_b = k_b k_u \frac{f_u}{\varepsilon_t + \varepsilon_u f_y}
\] (23)
or

\[
p_b = \frac{f_y}{f_u} = k_b k_u \frac{\varepsilon_u}{\varepsilon_t + \varepsilon_u}
\] (23a)

Stress-strain curves

Ultimate strength properties of the stress block were computed by equilib-
rium of forces and moments. By assuming that strain is distributed linearly
and that concrete stress is a function of strain only, Eq. (16) and (17) can be
used to derive complete stress-strain relationships for concrete in flexure.
The stress-strain curves obtained are shown at the left in Fig. 9. The curves
represent the average of values obtained by Eq. (16) and (17). Results ob-
tained by the two equations usually differed by about 2 percent, the maximum
difference being about 3.5 percent.

Stress-strain curves obtained in concentric compression tests of 6 x 12-in.
control cylinders are shown at the right of Fig. 9. Except for low concrete
strengths, the cylinders failed suddenly shortly after the maximum load was
reached. Lines related to the stiffness of the testing machine used are shown
in Fig. 9, and it is seen that the sudden failures always took place when the
cylinder stress-strain curve reached a slope equal to that of the machine
curve. Whitney's suggestion, that sudden failure of cylinders is related
to properties of the testing machine rather than to properties of the concrete,
is therefore substantiated.
Fig. 9—Concrete stress-strain relations
Comparison of the flexural and the cylinder stress-strain curves in Fig. 9 shows a striking similarity. The initial moduli of elasticity obtained from the two sets of curves are given in Fig. 10 as a function of cylinder strength. Though age at test affects the modulus-strength relationship, as would be expected, corresponding values from flexural and cylinder tests differ relatively little. The two moduli are compared directly in Fig. 11. The similarities shown in Fig. 9 to 11 indicate that the true general characteristics of stress-strain relationships for concrete in concentric compression are indeed applicable to flexure.

**Individual values of $k_1$, $k_2$, and $k_3$**

The maximum stresses obtained in flexural tests divided by the corresponding average cylinder strength (Fig. 9) give individual values of $k_1$; and $k_1$ may then be obtained from the measured values of $k_1k_2$. Values of $k_2$ and $k_3$ were obtained by Eq. (13). In this manner $k_1$, $k_2$, and $k_3$ were plotted in Fig. 12 as functions of cylinder strength $f'$. Straight lines define $k_1$ and $k_2$.

\[ k_1 = 0.94 - \frac{f'}{26,000} \]  \hspace{1cm} \quad (24)

\[ k_2 = 0.50 - \frac{f'}{80,000} \]  \hspace{1cm} \quad (25)

Eq. (18) and (24) then give

\[ k_3 = \frac{3000 + 0.35 f'}{3000 + 0.82 f' - \frac{f'}{26,000}} \]  \hspace{1cm} \quad (36)

Values of ultimate strength factors as given by Eq. (18) and (21) to (26) are summarized in Table 2 for convenient reference.
Phenomena of failure

Important observations regarding the failure phenomenon in concentric compression tests of concrete were made by Brandtzaeg through studies of volume strains

\[ e_v = e_l - 2e_t \]  

where \( e_v \) and \( e_l \) are numerical values of measured strains in longitudinal and transverse directions, respectively. At low loads he found that the volume of test prisms decreased roughly in proportion to the applied load. At 77 to 85 percent of the maximum load, however, volume started to increase at an increasing rate so that a volume expansion finally was present at failure. The load at which the derivative of volume strain with respect to load is zero was referred to as the "critical load." It is a fundamental assumption in Brandtzaeg's theory of failure for concrete that a progressive internal splitting is initiated at the critical load on minute sections scattered throughout the concrete mass.

For a small element at the compression face of a flexural member, a gradient of longitudinal strains is present, and strains in the two transverse directions are possibly not equal. Nevertheless, Eq. (27) may be used as an approximation, \( e_t \) being the transverse strain in the plane of the compression face.

### TABLE 2—ULTIMATE STRENGTH FACTORS

<table>
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<tr>
<th>( f_c' )</th>
<th>( f_u ) by Eq. (23a) for ( f_c = )</th>
<th>( k_{22} )</th>
<th>( k_k )</th>
<th>( k_{24} )</th>
<th>( k_e )</th>
<th>( k_h )</th>
<th>( k_{26} )</th>
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Fig. 11—Flexural versus compressive moduli of elasticity
In this manner Brandtzaeg found critical loads at 70 to 90 percent of the ultimate load for reinforced concrete beams and eccentrically loaded columns failing in compression. Similarly, Hognestad observed critical loads at 80 to 95 percent of the ultimate for eccentrically loaded columns.

Transverse strains were measured in the investigation reported herein by one 1-in. electric gage mounted on the compression face of each flexural specimen. In some cases a large piece of coarse aggregate was located adjacent to the gage and near the concrete surface. The transverse strain data were then not significant at high loads. Two examples of successful measurements are shown in Fig. 13. Points corresponding to critical loads are indicated on some flexural stress-strain curves in Fig. 9. The magnitude of the critical stress varies from 71 to 96 percent of the maximum stress.

For low concrete strengths, large deformations took place after the critical stress was reached and before failure took place (Fig. 9). For the 28-day specimen with w/c = 1.0, for instance, the strain corresponding to the critical stress was 21 percent of the ultimate strain. At the ultimate load, which was 58 percent over the critical load, therefore, 79 percent of the cross section was stressed beyond the critical stress. Even so, the first visible cracks, which were longitudinal tension cracks on the compression face of the specimen, were not observed before a load 99 percent of ultimate was reached. All low concrete strength specimens failed by crushing to a depth of 1/2 to 2/3 of the total depth of the cross section.
Behavior of the high-strength specimens differed somewhat. For the 28-day specimen with $w/c = 0.33$, the critical strain was 46 percent of the ultimate strain (Fig. 9), and the ultimate load was 66 percent over the critical load. The first cracks were seen at ultimate load, and crushing took place only to a depth of 1/5 to 1/4 of the total depth.

Concentrically loaded cylinder specimens will usually crack considerably if strained beyond the maximum load. In the flexural tests reported herein, a redistribution of stress may have occurred to less strained material closer to the neutral axis, so that strains greater than those corresponding to maximum stress were developed before cracks appeared. In tests of reinforced concrete beams, this important phenomenon was observed by A. N. Talbot in 1906, by O. Baumann in 1934, and by a number of investigators later.

**Structural concrete design**

The tests reported here show beyond reasonable doubt that the flexural stress-strain relation of concrete possesses a descending curve beyond the maximum stress. Near the ultimate load, therefore, the concrete stress distribution deviates considerably from the triangular distribution used in the straight-line theory. This is in complete accord with earlier indirect findings in which the shape of the stress block was deduced from observed behavior and strength of reinforced structural members. Therefore, these tests verify the reality and validity of the fundamental principles involved in ultimate strength flexural theories such as those presented by Whitney, Jensen, and others.

The numerical constants obtained, which characterize the properties of the stress block (Table 2), may be helpful in considering design values suit-
able for practical ultimate strength design. As far as the authors are aware, the present study represents the most complete investigation yet reported of measured properties of the stress block; but they are also fully aware that the tests cover only one type of aggregate, one maximum aggregate size, and one size of specimen without compression reinforcement. Nevertheless, it is believed that the tests have disclosed both general principles and quantitative data of some significance in the field of ultimate strength design.

To make definite design recommendations, however, strong consideration must be given also to tests of reinforced concrete beams and columns, to simplification of routine design methods, to construction practices, to overload factors, and to many other matters beyond the scope of the present paper. Such recommendations are available in the report of Committee 327, Ultimate Load Design, a joint ACI-ASCE committee.

SUMMARY

This investigation was conducted with the primary objective of developing a test method leading to an improved quantitative understanding of concrete stress distribution in flexure. In other words, it was our goal to strengthen knowledge regarding the fundamental principles involved in ultimate strength flexural theories.

A study of previous experimental investigations regarding the stress block revealed that, though many test methods have been tried, very limited direct test data are available. On the other hand, considerable information regarding the stress block has been derived indirectly from strength and behavior observed in numerous previous tests of reinforced structural members.

A test method was developed in which a 5 x 8-in. unreinforced concrete section was loaded with an eccentricity that was varied during each test in such a manner that the neutral axis remained at a face of the section throughout the test to failure. The average compressive stress in the concrete then always equaled the total axial load divided by the section area, and the centroid of the stress block coincided with the eccentricity of the total applied load. Furthermore, by numerical differentiation the flexural stress-strain relationship of the concrete was determined from zero load to failure.

Such tests were made for concretes with w/c ratios of 1.0, 0.67, 0.50, 0.40, and 0.33 at test ages of 7, 14, 28, and 90 days. A striking similarity was found between flexural stress-strain relations and those obtained in concentric compression tests of 6 x 12-in. cylinders. In both cases a descending curve was observed beyond the maximum stress. The numerical values obtained, which characterize the properties of the stress block, are in general agreement with values derived earlier from tests of reinforced structural members.

The tests reported herein, therefore, strengthen our knowledge regarding the stress block in flexure, and the test data obtained demonstrate the reality and validity of the fundamental plasticity concepts involved in ultimate strength flexural theories for structural concrete such as those presented by C. S. Whitney, V. P. Jeneen, and others.
REFERENCES


22. Jensen, V. P., "Ultimate Strength of Reinforced Concrete Beams as Related to the Plasticity Ratio of Concrete," *Bulletin No. 345*, University of Illinois Engineering Experiment Station, Urbana, June 1943, 60 pp.


DISCUSSION

of the paper

CONCRETE STRESS DISTRIBUTION
IN ULTIMATE STRENGTH DESIGN


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The extensive tests carried out by the authors represent a valuable investigation of the stress distribution in compression of concretes of different strength properties. The method of testing was ingenious and the introduction of the "critical" stress in the stress-strain diagrams for compression and flexure was interesting. This critical stress appears to be important to ensure safe values when the compressive resistance of the concrete is of direct influence upon failure, as long as the margin between actual test results and calculation is not too great. With regard to tensile failure, however, there is no need for an additional factor of safety between actual failure load and calculated load, particularly if the reinforcement is well bonded. This will result in a good crack distribution with rather fine cracks in which necking of the steel will be less than in wide cracks; consequently there will be in many cases an excess of strength as compared with the calculated value based on the yield point. This can be concluded from a great number of test results showing higher ultimate bending moments than calculated. It may be pointed out that so far nobody has been able to measure the strain of steel within a narrow crack, nor even over a correspondingly short gage length.

In view of these considerations, it seems of interest to compare the data given in the paper with actual test results. The writer has investigated this question on numerous tests, limited to high-strength materials in ordinary and prestressed concrete, and an approximation, Eq. (a), has given satisfactory and safe values.

\[ R = \frac{M_{ult}}{bd} f'_c = q (1 - 0.5 q) \]

In this equation a limit of \( k_2 = 0.5 \) has been taken into account for a rectangular stress block with \( k_2/k_3 = 0.5 \). When comparing Eq. (a) with the authors' Eq. (3) and Table 2, we find that the ratio \( k_2/k_3 \) agrees approximately only for a concrete strength of 1000 psi with the factor of 0.5, when
it is 0.48, while with higher concrete strengths it increases up to 0.67 (for $f_0' = 8000$ psi), and thus the ultimate bending moment would become less than the values obtained from Eq. (a). The actual test values investigated by the writer in reference 1 were higher than those computed according to the writer’s Eq. (a). Consequently he showed in reference 2 that this excess in strength, apparently due to reduced necking of the steel in the cracks, can be taken into account by considering an equivalent cooperation of the concrete in the tensile zone, based on his suggestion in the discussion to Cox’s paper.  

$$R = M_{ut}/bd'f_0' = \left[ q(1 - 0.5 q) - f_0'[t/d - 1] q - 0.5 (t/d)^2 \right]/(1 + f_0') \ldots (b)$$

In Eq. (b), $f_0$ is the ratio of the equivalent concrete tensile stress, extending uniformly over the depth $(t - c)$, to the cylinder strength $f_0'$. This equation gives higher $R$ values in the range in which failure occurs due to yielding of the steel and thus agrees better with the test results.

For balanced design, $q_b$ becomes 0.5 according to Eq. (a) and results in $R_b = 0.375$; according to Eq. (b) $R_b$ is only slightly increased to 0.387 for the same $q_b = 0.5$, for a factor $f_1 = 0.05$ and the ratio $t/d = 1.1$, which results generally in

$$R = M_{ut}/bd'f_0' = 0.029 + 0.957q - 0.481q^2 \ldots \ldots \ldots \ldots \ldots (c)$$

In the following, Eq. (3), (23), and (23a) are investigated for concrete of strength $f_0' = 6000$ psi. According to Table 2 for this strength the following data apply: $k_0/k_1 = 0.65, k_1k_2 = 0.65$, and $e = 0.0031$. Thus the equation equivalent to Eq. (a) becomes

$$R = M_{ut}/bd'f_0' = q(1 - 0.65q) \ldots \ldots \ldots \ldots \ldots (d)$$

and much lower results are obtained than from Eq. (a), which has given safe values at least for high-strength materials. In the following table the authors’ data given for $f_0' = 6000$ psi are evaluated for $E = 30 \times 10^6$ psi for various steel stresses, considering that high-strength steel is used and prestressing may be applied.

<table>
<thead>
<tr>
<th>$f_y$ ksi</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
<th>100</th>
<th>150</th>
<th>200</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_0$</td>
<td>0.700</td>
<td>0.600</td>
<td>0.608</td>
<td>0.571</td>
<td>0.557</td>
<td>0.508</td>
<td>0.482</td>
<td>0.383</td>
<td>0.319</td>
</tr>
<tr>
<td>$q_b$, percent</td>
<td>0.84</td>
<td>0.73</td>
<td>0.65</td>
<td>0.57</td>
<td>0.52</td>
<td>0.48</td>
<td>0.44</td>
<td>0.39</td>
<td>0.33</td>
</tr>
<tr>
<td>$q_b$</td>
<td>0.555</td>
<td>0.422</td>
<td>0.365</td>
<td>0.371</td>
<td>0.349</td>
<td>0.329</td>
<td>0.313</td>
<td>0.249</td>
<td>0.206</td>
</tr>
<tr>
<td>$R_b$</td>
<td>0.331</td>
<td>0.300</td>
<td>0.294</td>
<td>0.282</td>
<td>0.270</td>
<td>0.250</td>
<td>0.240</td>
<td>0.209</td>
<td>0.178</td>
</tr>
</tbody>
</table>

These values may be compared with some test results investigated by the writer.  

Beam No. 24, Table IX, of the writer’s tests with high-strength materials in Vienna (1935) resulted in a relatively higher value $R = 0.246$ for $f_y = 91$ ksi and a cube strength of 8500 psi, corresponding to $f_0' = 6000$ psi,
the percentage being only 1.47 percent, which is much below \( p_b = 2.19 \) percent computed in the above table for \( f_y = 90 \) ksi.

Another comparison may be made with regard to prestressed concrete, and beams No. 10-18 of Tables I and III of reference 1 will be considered. These relate to tests carried out by Stott, Leeds, using concrete of a cube strength of 8250 psi, approximately \( f'_c = 6000 \) psi, the strength of the wires being 208 ksi and the initial prestress varying between 171 and 182 ksi. The \( R \) values obtained for these tests of percentages between 0.77 and 1.08 percent varied between 0.352 and 0.425; however, a certain amount has to be deducted, since 9 to 11 tensioned wires 0.08 in diameter were placed in the compressive zone, 1/2 in. below top surface. If the ultimate bending moment is reduced by the resistance of the compressive reinforcement, computed for a steel stress of 100 ksi, the \( R \) values are reduced to 0.316-0.386. When comparing these values with those shown in the table on p. 1306, first the appropriate steel stress \( f_y \) must be selected.

There are different views about the elongation \( \varepsilon_c \) to be considered for tensioned steel in prestressed concrete. According to one school of thought, only the difference between the elongation of the steel at failure and that of the effective prestress should be taken into account. According to the other school, the entire elongation applies, whether the steel is tensioned or not. This is based on the assumption that, when failure approaches, the prestress relaxes considerably and may even become zero in a crack. If in the special case an effective prestress of 140 ksi and a yield point of 220 ksi is considered, resulting in a difference of 80 ksi, then according to the above table, \( R_b = 0.270 \) and \( p_b = 2.2 \) percent would apply, whereas in fact \( R \) values 0.315 to 0.386 were obtained for percentages much below 2.2 percent. This discrepancy would become even much more pronounced, if instead of the difference of elongation the entire elongation of the wire were taken into account.

It would appear that neither of the two assumptions mentioned above, related to the ultimate elongation of the steel, is correct but that an intermediate solution applies, and the writer would be glad to hear the authors' opinion on this question particularly in view of the above examples. As already mentioned, insufficient data are available about the actual elongation of the steel in a crack, which solely influences failure conditions if the beam is under-reinforced. The efficiency of the tensile resistance will, in fact, mainly depend on the bond resistance, which is less easily obtained with untensioned high-strength steel than with prestressed steel, as Janney has clearly shown.

It may be pointed out that the results obtained by the authors and summarized in Table 2 give important data about the compressive resistance of concrete. However, their application to the design of reinforced or prestressed concrete beams will have to be investigated by further tests in which the actual resistance of the tensile reinforcement is taken into account and not based on the assumed elongation at failure. These remarks refer solely to high-strength materials. For ordinary reinforced concrete with mild steel
the data given in Table 2 will most certainly give satisfactory results for ultimate load design.

REFERENCES


By A. J. ASHDOWN*

The method of determining the stress distribution in the compression zone of a concrete beam put forward by the authors represents a distinct advance in experimentation for this purpose. The results are not complicated by the interaction between concrete and steel, such as occurs in beams. The differences between these test results and those obtained from beams are quite wide as revealed by Fig. 8(a), and in my opinion this is due to insufficient appreciation of what occurs between the steel and concrete in beams, before and after cracking. The usual assumption of perfect bond does not occur; the concrete drags on the steel (especially noticeable with deformed bars), and between cracks lowers the stress in the steel, or puts it into a relative compression. Close to the support the steel may actually be in compression; if this were so, the zero for tensile stress would be lowered. This would partly account for the so-called "hyperstrength" of reinforced concrete beams. This means that, although the concrete has cracked, it is still, partly if not wholly, assisting in resisting bending through the steel between the cracks.

It may be noted, that in the authors' test piece, the compressed length is quite short compared with a beam, which would require an effective depth of at least double that of the specimen for an equivalent compressed area, so that for a third-point loaded beam, one should expect that the ultimate compressive strength values on the much longer length to be rather lower than those obtained in these tests, even taking into consideration the narrow width employed.

The wide scatter of values for the modulus of elasticity of concrete is well known. I use a simple formula for the initial modulus, $E_c = 52,000 \sqrt{f_{cu}}$ or $55,300 \sqrt{f_{ct}}$, for cube strength or cylinder strength, respectively, and which lies between Jensen's and the proposed formula in Fig. 10.

It is gratifying to note that the strain profiles are essentially linear.

It is altogether an excellent and stimulating paper.

At Imperial College, during the last 10 years, we have carried out a large number of beam tests and obtained results which agree closely with those shown in Fig. 6. We have adopted 0.4 as a safe value of $k_2$. Using strain gages to determine the position of the neutral axis at failure, we have determined the average compressive stress in terms of the cube strength $c_u$ and found that safe values are 0.6 $c_u$ for concretes weaker than about 4000 psi and 0.5 $c_u$ for concretes stronger than about 4000 psi. Since cylinder strength is about 0.8 cube strength, these values agree well with the $k_1k_2$ values of Fig. 6.

We have also found that 0.002 is a safe minimum value of the concrete strain at which maximum strength occurs, as shown in Fig. 2. We agree, too, with the assumption of a linear distribution of strain, except at wide cracks when the ratio of the steel strain to the virtual concrete strain equals $F$. The value of $F$ varies from about 0.8 to 1 according to bond conditions. The strain values at failure determine the position of the neutral axis, and the moment of resistance of a beam is then easily obtained from the above safe values. In bonded prestressed beams, the steel strain is assumed to be the total strain minus the strain due to prestress.

Some further tests have been carried out on the bending simulation machine referred to in the paper and agree well with the beam tests. Precise determination of the shape of the compressive stress distribution is a slow procedure, but it has been obtained. It has become, however, mainly of academic interest, since safe values of the average compressive stress and the position of the center of compression obtained from many beam tests provides sufficient information for design purposes. The position of the center of compression is determined from the lever arm value, the latter being obtained from the measured total steel tension and the applied bending moment.

This excellent paper gives rise to some reflections, which might be of general interest. The majority of reinforced concrete beams are under-reinforced and in those cases the difference in calculated ultimate bending strength according to conventional calculations and to ultimate strength design is negligible. In over-reinforced cross sections, such as columns and some types of beams with high quality steel, the compressive area takes up a rather large part of the whole cross section. Sustained loads can in that case give rise to appreciable creep in the concrete. In the usual beams the actual creep deformations will have comparatively low values, because a lowering of the neutral axis will decrease the concrete stresses to a high degree. This will not
happen to the same extent in over-reinforced cross sections. The question is: should the creep deformations be included in the ultimate concrete strain or not? If we take an extreme case of a cross section having a sustained highest compressive stress of about 2000 psi, the corresponding elastic strain is of the order 0.0004 and the creep deformation might be of the order 0.0012, which is 40 percent of the value of the ultimate strain recommended by the authors.

The average compressive stress at compression failure depends, as the authors have pointed out, quite a lot on the effect of transverse strains. Appropriately placed stirrups thus can to a certain degree increase the ultimate compressive force. A similar effect can be obtained if the compressive area of the cross section is sustained by concrete with low stresses. This can especially be the fact, when the distance to the neutral axis is small. I have found with a 12 in. high cross section (cube strength 9000 psi) that the height of the
compressive stress in the concrete was higher than the cube strength. On the other hand, the dimensions of the test specimens used by the authors correspond to the compressive area of an usual beam with a depth of about 60 in. There may be more favorable results obtained by such small beams as joists.

Finally I would like to mention the obtained values of the initial modulus of elasticity. Starting from the point of view, that in this case these values ought to depend on one quality factor and one time factor, I have arranged the values in flexure, measured from Fig. 10, as a function of the 10th logarithm of the age of the concrete and of the cement-water ratio, namely

$$F(t, w/c) = (0.24 + 1.43 \log t + 0.75 c/w) 10^6$$

where $t =$ age of concrete in days. As seen from Fig. A, the result seems to be significant as a first approximation.

By HENRY J. COWAN*

The paper is an outstanding contribution toward the design of reinforced and prestressed concrete for ultimate strength. In the past it has been necessary to assume a shape for the concrete stress distribution, and to prove the accuracy of the assumption by comparing the predictions of the resulting theory with the failing loads of test beams. We have now for the first time extensive experimental data of the shape of the stress-strain diagram itself.

The most significant features of results presented are the relatively high ultimate strains and the descending portion of the stress-strain diagram. Neither can be reconciled with mathematical theories of the inelastic deformation of concrete which have been published. Glanville's experiments in 1930 established the existence of inelastic strains which were proportional to the applied stress. As a result, a number of rheological models for the deformation of concrete were proposed consisting of viscous dashpots and elastic springs. While these models satisfactorily reproduce the long-term deformation of concrete at moderate loads, they fail to explain the high ultimate strains and the descending portion of the stress-strain diagram.

Freudenthal has suggested that the departure of the concrete stress-strain diagram from visco-elastic behavior is due to breakdown of the adhesion between the cement and the aggregate. This presumes the formation of micro-cracks above a certain limit. The presence of minute cracks at about 60 percent of the ultimate load has in fact been observed by Berg, using an optical method, and by Jones, using an ultrasonic method; their presence has also been inferred from tensile tests by Blakey and Beresford.

These considerations lead to the statistical concept of a concrete specimen made up of a number of elements with strengths distributed over a specified range. This may be represented in a mechanical model by a number of springs rupturing progressively with increasing strain, and thereafter ceasing to contribute to the load resisting capacity of the model. Williams and Kloot, working on the behavior of timber specimens in tension, have pro-

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†The model proposed by A. D. Ross ["Creep and Shrinkage in Plain, Reinforced, and Prestressed Concrete," Journal, Institution of Civil Engineers (London), V. 21, 1944, pp. 38-57] may serve as an example.
duced a mathematical theory for a model of this type. Choosing suitable parameters, a stress-strain curve obtained from this theory is shown in Fig. B. The theoretical curve obviously proceeds to an infinite strain at a very small stress, since in a statistical theory an occasional element may remain undamaged, and it is necessary in practice to stipulate a limiting ultimate strain. With this reservation, however, the agreement with the experimental data presented in this paper is evidently satisfactory, and it should be practicable to determine parameters to fit various types of experimental curves.

The mechanical model shown in Fig. C has therefore been constructed to reproduce all the known features of the inelastic deformation of concrete. It consists of two series of viscous dashpots and elastic springs in parallel, the dashpot of one of the units being blocked by nonreturn valves; and a series of brittle springs in parallel which behave elastically up to failure, the failing loads of these brittle springs being distributed over a range. When this model is deformed by a moderate load, so that the brittle springs are not damaged, it exhibits an immediate elastic deformation proportional to the applied load, and a progressive inelastic deformation at a decreasing rate, the total inelastic deformation over an infinite period being proportional to the applied load. When the model is unloaded the procedure is reversed; but due to the blocking of the second dashpot some of the inelastic deformation is not recovered. When the model is loaded sufficiently rapidly to produce negligible viscous deformation of the dashpots, it deforms elastically until the fracturing load...
of the weakest brittle spring is reached, and the subsequent inelastic deformation is irreversible. When the model is deformed sufficiently slowly to allow for appreciable deformation of the dashpots and the load is carried at the same time above the fracturing load of the weaker brittle springs, then both types of inelastic deformation are reproduced.

It may be seen that the repeated loading of reinforced concrete structures at high loads (i.e., at loads above the working loads but below the failing loads under rapid loading) present a problem that must be investigated before ultimate strength methods can be applied to reinforced concrete structures with perfect confidence. It has been observed that inelastic deformation due to creep, by ironing out the peak moments in a redundant structure over long periods of time, increases the load bearing capacity of the concrete structure. It has similarly been observed that micro-crack formation, by extending the stress-strain diagram and increasing the ultimate strain, increases the load bearing capacity of the concrete structure loaded continuously to destruction, as compared with a structure made from a similar elastic material failing in a brittle manner at the maximum stress. If it is, however, accepted that the declining portion of the stress-strain diagram is due to irreversible micro-crack formation, then the increase in strength may not hold for repeated high loads.

The outstanding success of the authors in developing an experimental technique for determining the complete stress-strain diagram under rapid loading holds out hope for the solution of the problem of repeated loading.

REFERENCES


By HOMER M. HADLEY†

The series of tests reported in this paper is notable in its abandonment of the standard 6 x 12-in. concrete cylinder as the means of demonstrating the immediate plasticity of concrete in response to overload. Instead of the

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*In practice some of the micro-cracks would probably close up due to autogenous healing, if sufficient time is allowed.
†Hadley & Hadley, Consulting Engineers, Seattle, Wash.
cylinder there is substituted the channel-shaped frame, simultaneously subjected to direct stress and bending, and a new foundation for ultimate strength design is presented.

The results of tests of these frames and of companion cylinders is presented in Fig. 9. There is a marked similarity in the stress-strain curves of two types of specimen. Perhaps the most significant feature of the curves is their limited extent; except for those of lowest strength, none reach far beyond strains of 0.003 in. per in. "Where is the rose of yesteryear?": those strains of 0.009 and 0.010 in. per in. that once graced and adorned the outer reaches of "the plastic range"? As a matter of fact, between the 0.002 strains of the elastic range and the 0.003 to 0.0035 boundaries of the elastic range, not much space is left for plasticity.

It is noted that about 15 min was given to the testing of each frame. This permits a more gradual application of load and more gradual adjustment to load on the part of frame than would be possible in a briefer time. The writer is of the opinion that the sudden application of a small increment of load to any of these frames that had attained a strain of 0.0025 in. per in. would have abruptly terminated both the curve and the frame at that point. The weakness and "come-apartness" of concrete so strained in a brief period of time has not been lost by changing the form of the specimen. The identity of those frames shown in Fig. 5 is not disclosed but the shattered condition they are in was no sudden development preceded by a relatively lengthy period of plastic integrity.

Not the least unsatisfactory feature of this ultimate strength theory is the fact that there is no definite and decisive physical boundary to limit its supposed plastic range. The elastic theory has the maximum cylinder strength which is something positive and specific and determinable. This theory has neither a definite strain, nor a definite state of cracking and shattering, nor a definite strength after a definite time of applying load at a definite rate. Its physical basis is vague and uncertain and indefinite. We have been offered the Saliger speculation, the curves extending to strains of 0.009 and 0.010 without word of the concrete’s coming apart, and others. It is to be hoped that the authors will state their recommendations respecting the outer limits of the plastic range.

Fig. 2, titled "Tests of 3 x 6-in. concrete cylinders—U. S. Bureau of Reclamation," has a familiar look. On comparison the curves are found to be identical with those of Fig. 3, ACI Journal, Oct. 1952, Proc. V. 49, p. 122, where they appear in Mr. Hognestad’s paper, "Inelastic Behavior in Tests of Eccentrically Loaded Short Reinforced Concrete Columns." The chief difference between these two figures is that the curves are now designated, top to bottom: '41 days, 13 days, 9 days.” These designations now definitely identify the curves with those bearing the same descriptions in Fig. 8 of Laboratory Report No. SP-12, Mar. 7, 1947, U. S. Bureau of Reclamation, wherein we learn that "each curve represents the loading of a single 3 x 6-in. cylinder.” These then are the cylinders whose stress-strain relationships
for concrete in compression Professor Hognestad believed applicable to flexure and which he then took as the basis of "the present analysis," i.e., that of 1952.

Under the circumstances prevailing it is scarcely to be wondered at that L. P. Brice's experimental efforts\textsuperscript{17} to determine flexural stress-distribution in a beam receive the somewhat hurried brush-off that they do in this paper "... a straight-line stress-distribution was found, even at high loads. The principal weakness of this experimental method is the unknown extent to which the introduction of the steel plates locally changes the stress conditions in the concrete."

It is far from clear to the writer that the introduction of steel plates in Brice's beam did actually change the stress condition in the concrete appreciably. But if it did, why of all things did the stress pattern so perversely revert to one with straight-line variation, after all the labored efforts that have been made to have it a "plastic" pattern? In any event it is probably well that this beam be described in some detail so that "its principal weakness" can be more widely evaluated.

Brice's test beam is a novel and most interesting one. Its dimensions are slight: $7 \times 16 \times 150$ cm. It was tested on a 130-cm span with equal, approximately third-point loads. At midspan a stack of eight steel plates, 14 cm long, replaced that length of concrete in the upper two-thirds of the section and an open gap or hole, about 8 cm long, was created in the concrete beneath them, across which opening the reinforcing bar extended. The eight plates, 1 cm thick, were not in contact with one another but were separated by spaces of 0.413 cm. These plates were placed and the concrete was cast against their ends and against the little forms which excluded it from the central open gap portion beneath them. In plan view these plates were hourglass shaped, of full beam width, 7 cm, at their ends narrowing to a section 1 cm wide, at the middle, which reduced section extended for a length of 4.5 cm. Each plate had two small projecting prongs at each end for anchoring it into the concrete and holding it in position.

Thus, to repeat, there is created a concrete beam with reinforcement continuous from end to end, with a midspan zone of constant moment and zero shear. At midspan there is a short length from which concrete is completely excluded, the gap being crossed by the reinforcing bar at the bottom and at the top by a set of steel plates about $\frac{3}{8}$ in. thick having about $\frac{3}{8}$ in. gap between them. The ends of these plates bear against the full width of the concrete but each plate reduces, as described, to a central section $\frac{3}{8}$ in. square by $1\frac{3}{4}$ in. long, completely out of contact with its neighbors.

Such a beam appears innocent enough. It is only the concrete, pressing against the steel plates on their end surfaces that produces the recorded deformations in the steel. The steel, of course, reacts against the concrete but it is the pressure, the force, exerted by the concrete, not the strains and deformations of the concrete, that produce the deformations in the steel and these steel deformations are not so large as to involve any question as to their in-
interpretation: they are purely elastic. Now it is true that the plates show a straight-line stress distribution pattern "even at high loads," but even so that fact is not, per se, discreditable to them, does not convincingly demonstrate that they have "locally changed stress conditions in the concrete." All they have done is conflict flagrantly with the theory which our authors seek to establish. Another possible view is that Brice has developed a simple, direct, and most ingenious method of determining what the stresses in concrete actually are, quite independent of concrete deformations and the interpretation thereof. The installation of a set of Brice plates in the midsection of a concrete channel frame might present its difficulties but with the ordinary beam there is little or no trouble to be foreseen. It is to be hoped, therefore, that this promising field of investigation will be thoroughly and open-mindedly explored, both by the authors and by others.

Some 25 years ago Slater and Lyse found the flexural strength of concrete, based on the straight-line theory, in sufficiently reinforced concrete beams to be 67 percent higher than that of 6 x 12-in. cylinders with 2000-psi concrete; 50 percent higher with 3000-psi and stronger concrete. Nearly 30 years ago Gonnerman's tests showed that the strength of concrete in 6 in. diameter cylinders varied inversely with the height of the cylinder, in a fashion not dissimilar to the tones of a trombone under varying degrees of extension. Just why the compressive strength of the 12 in. high cylinder should be expected to be the exact or even approximate measure of the flexural strength of its concrete rather than that the strength of a 6 in. high cylinder should, say, or the strength of some other height of cylinder, cannot rationally be explained but the wide prevalence of that belief is not to be questioned. Consequently, if the maximum assumed value of stress in flexure is to be no more than that of the cylinder, the only way the beam's flexural capacity can be explained is by scrapping the straight-line theory and providing a substitute theory that activates the region in the vicinity of the neutral axis which the straight-line theory excludes from consideration. On the other hand, if one can entertain the thought of a flexural unit stress greater than the compressive unit strength of the 12 in. high cylinder by a Slater-Lyse amount, then the straight-line theory works admirably and agrees with the beam performance. The straight-line theory being the one with which everyone is familiar, it seems much simpler and better to change the value of $f_c$ than to scrap the straight-line theory altogether and to adopt a new one founded on the wonder tale of concrete's instant plasticity. Therefore despite the addition of a new section on ultimate strength design in the ACI Building Code and other "party line" efforts, it may be advisable to hold back a bit pending a proper investigation rather than an arbitrary rejection of Brice's findings, which, as stated, showed a straight-line variation of stress "even at high loads."

Would it not, at this stage of things, be a most amazing development if additional Brice-beams should reveal naught but straight-line stress variation?
The investigation by the authors forms a conclusive contribution to the ultimate strength theory, which has been achieved by means of an original and ingenious method. I would like to make a few comments to the authors' remark on p. 477 that "the tests cover only one type of aggregate, one maximum aggregate size, and one size of specimen without compression reinforcement."

I compared the test data given in the paper with results obtained elsewhere. In the USSR, for instance, eccentrically loaded columns of rectangular and I-shape sections have been tested up to considerable dimensions; the largest specimen had a section of 80 x 160 cm (about 32 x 64 in.). Comparative tests were carried out on columns, the sections of which had areas twice to 64 times smaller. Other investigations have been carried out in several European countries, different types of aggregates and cements have been used, and different testing arrangements applied.

All the accumulated evidence leads incontestably to the conclusion that the stress-strain relationship for concrete in flexure is nonlinear and has to be represented by curves similar to those shown in Fig. 9. I think, therefore, the authors' investigation gives qualitative information of fairly general validity.

Let us now compare the quantitative test data arrived at in the paper. Eq. (3) may be rewritten in the form

\[ M_{\text{ult}} = A_s f_y d \left( 1 - \frac{k_2}{k_1} \frac{f_y}{f'B} \right) \]  

(a)

The value of \( k_2/k_1k_3 \) depends on the concrete strength; according to Table 2 it varied between 0.48 and 0.67. The difference is significant.

In Europe, we consider it in practical computations in the following way. We substitute a rectangular stress block for the actual one. To obtain the correct moment \( M_{\text{ult}} \) given by Eq. (a), both the actual and the substituting stress block must have the same volume and give the same line of action of the total compression \( C \) (see Fig. 1). This means for a rectangular section that the substituting rectangular stress area has a depth of \( 2k_c \). Denoting with \( f'_{\text{flex}} \) the assumed uniform stress in the pressure zone (the so-called flexural strength of concrete) we obtain, instead of Eq. (a), the expression

\[ M_{\text{ult}} = A_s f_y d \left( 1 - \frac{1}{2} \frac{f_y}{f'_{\text{flex}}} \right) \]  

(b)

Both Eq. (a) and (b) yield the same result, if

\[ \frac{k_2}{k_1 k_3} = \frac{f'_{\text{flex}}}{2 f'_{\text{flex}}} \]  

(c)

The standard specifications of these European countries, which have adopted the ultimate strength design, determine simply the value of \( f'_{\text{flex}} \) for

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each category of concrete so that it corresponds to the test value of $k_2/k_3$ according to Eq. (c). The category of concrete is defined by the 20-cm cube strength of concrete.

In this country, for instance, the standard values of $f'_{flz}$ (given in the 1955 specifications) have been determined according to the following formula, which was suggested by the discussor with regard to previous test evidence:

$$f'_{flz} = \frac{1435 f'_{cub}}{1490 + f'_{cub}} \quad \dot{\ldots} \quad (d)$$

Here, $f'_{cub}$ is the 20-cm cube strength of concrete in kg per sq cm. To compare the test results, from which Eq. (d) has been derived, with those given by the authors, we have to take into account that 1 psi = 0.0703 kg per sq cm and that $f'_{cub}$ is regarded as being equal to 1.05 $f'_c$.

Using these relations and substituting into Eq. (c), we obtain the values of $k_2/k_3$ corresponding to the Middle European tests. The results are given in the third column of Table A.

Evidently, both groups of tests lead to practically the same conclusions; the greatest difference between both respective numbers is 3 to 4 percent of the authors' values.

Russian standard specifications of 1948 give values of the strength $f'_{flz}$ which are 5 to 7 percent higher than those calculated by means of Eq. (d). This has but an insignificant influence on the resulting ultimate moment given by Eq. (b); the difference against Eq. (a) does not reach 3 percent.

It can therefore be stated that the findings of the authors are also quantitatively in accord with European investigations. The small variations can easily be explained by local influences.

In my opinion the conclusions arrived at in the paper may be considered as a definitive answer to the question of the concrete stress distribution.

REFERENCES

3. GOST 4286-48 (USSR standard specifications) and NiTU-3-48 (Russian norms and regulations), Moscow, 1948 (in Russian).

By J. M. PRENTIS* 

On p. 462 the authors point out that their method has advantages over the methods of analyzing beam test data proposed by Prentis, Hamman, and Lee for it avoids the complications introduced by the presence of the rein-

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forcement and the concrete tensile stresses. There is a further advantage which, in some respects, is more important than the two already mentioned, for the difficulty of measuring the steel strain can be overcome* and the tensile stresses are generally insignificant in their effect. There are, however, inherent difficulties in measuring accurately the concrete strains in a beam, due to the finite and often irregular spacing and form of the tensile cracks. It is necessary to measure average strain readings over a 6- or 8-in. gage length to determine the maximum concrete strain (\(\varepsilon_c\)) and the position of the neutral axis. Thus, the section of failure \(\varepsilon_c\) is somewhat greater than the mean value and the depth of the compression zone is less than that indicated by the strain gages. Furthermore, it is difficult to estimate the extent to which finite spacing of cracks upset distribution of stress above neutral axis.

In deliberately ensuring that no tensile stresses are generated the authors avoid these difficulties. The writer would like to congratulate the authors on the ingenious way in which they have done this.

However, having isolated the problem of determining the concrete stress from the complications due to reinforcement, concrete tensile stresses, and crack spacing there appears to be a logical difficulty in applying the results to normal reinforced concrete beams where all these factors are present. The writer would like to suggest that it is desirable that further tests be carried out in which tensile cracks are allowed to form.

A comparison of the results from tests on pairs of identical specimens in which one of a pair is tested without cracks while its mate is tested with cracks would demonstrate the effect to which the crack spacing and presence of tensile stresses upset the analysis. The writer must confess to a personal interest here; such tests would indicate the extent to which the beam analyses of Prentis, Hamman, and Lee are well founded.

One of the basic assumptions of all the methods is that concrete stress is a function of strain only, time and stress history effects being negligible. The validity of this is questionable but it appears that the authors could use their test methods to verify it. It would be possible to test identical specimens at different rates of loading; or alternatively, single specimens could be tested so that the strain at one edge was held at, say, twice the strain at the other and so strained twice as fast. If time effects are negligible then the resultant stress-strain curves from such tests would be the same whatever the rate of straining.

Beam tests by the writer have provided evidence substantiating this assumption. These tests are referred to in a paper by the writer† but the relevant points may be mentioned here. A number of beams were tested under four-point loading to give a central span subjected to a uniform bending moment. With each beam it was possible to measure strains at three sections along the length of the central span. Due to irregular cracking the compressive strain in the concrete and the position of the neutral axis varied between

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sections. Using the writer's method of analyzing the data the stress distribution was drawn for each of the three sections. Although the strain distribution, and hence the rate of straining, differed for each section the stress-strain curves were found to be the same. The time taken for testing each beam was 4 to 5 hr. Had time effects not been negligible this would have resulted in a different stress-strain curve for each section.

By E. ROSENBLUETH*

The paper constitutes an important contribution to the theory of ultimate strength design. This discussion should not detract from the main conclusion of the paper, namely, that the proposed curved distribution of stresses in concrete leads to fairly accurate prediction of flexural capacity in prismatic members of rectangular cross section. Nevertheless, it does warn against unwarranted extrapolation and neglect of time effects.

Conclusions derived in the paper may be grouped in two categories. The first concerns stress distribution in prisms loaded similarly to those reported in the paper. The methods used for computing stress-strain curves in the prisms are objectionable principally because of the assumption that concrete stress is a function of strain only and does not depend on the history or rate of loading. It is well established that stress-strain relations in concrete are strongly time sensitive. Now, the strain rate in the prisms varies approximately in proportion to the distance from the neutral axis, where it is zero. Admitting a linear variation of $\epsilon$, one may write

$$\dot{\epsilon} = \frac{a}{c} x$$

where the dot denotes derivative with respect to time. The effects of such a pronounced variation in strain rate should certainly not be neglected.

To gain some idea as to effects of the variable strain rate, it will be assumed that concrete may be represented as a certain combination of Maxwell and Kelvin bodies. That is, the behavior of each element of concrete will be idealized as that of a system of linear springs and viscous dampers. Under these conditions, extrapolation to $\dot{\epsilon} = \infty$ from direct compression tests performed at different rates should give a linear relation between stress and strain. Glanville,† as well as several other investigators, has shown conclusively that this is nearly the case, at least up to stresses equal to 0.8 $f'_c$. Admitting the validity of the proposed idealization in the range $0 \leq f \leq 0.8 f'_c$, the stress-strain relation can be put in the form

$$f = \phi(\epsilon)$$

where $\phi$ depends on the strain rate and strain history.

Again, assuming linear strain distribution in the prisms, it is simple to show that

$$f = \phi' \left( \frac{\epsilon}{c} \right)$$

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and the stress distribution is linear at all times up to \( f_0 = 0.8 f' \). From Eq. (A3)

\[
f_0 = \frac{1}{c} \int_0^c f_0 \, dx - \frac{1}{2} \phi'(\epsilon_t)
\]

and from Eq. (16)

\[
f_0 = \frac{1}{2} \left[ \phi(\epsilon_t) + \phi'(\epsilon_t) \epsilon_t \right]
\]

where the prime denotes derivative with respect to \( \epsilon_t \). Similarly, from Eq. (17)

\[
f_0 = \frac{1}{3} \left[ 2\phi(\epsilon_t) + \phi'(\epsilon_t) \epsilon_t \right]
\]

The difference between values of \( f_0 \) computed from Eq. (A4) and (A5) is

\[
D = \frac{1}{6} \left[ \phi(\epsilon_t) - \phi'(\epsilon_t) \epsilon_t \right]
\]

It is true that \( D = 0 \) only when \( \phi/\epsilon_t \) is constant, which corresponds to no internal damping. However, \( D/f_0 \) is quite small in practice. For example, in the range covered by the tests one may take, in first approximation, each concrete element idealized as a simple spring and dashpot in series. Then, for a constant strain rate \( \dot{\epsilon} = v \)

\[
\phi = v \gamma \left( 1 - e^{-\frac{E \epsilon_t}{v \gamma}} \right)
\]

where \( \gamma \) and \( E \) represent the damper and spring constants, respectively. Substituting in Eq. (A4) to (A6) and assuming \( v_c \gamma = 2.6 E \epsilon_t \) at \( f_0 = 0.8 f'_C \), which is representative of the curves in Fig. 9, one obtains \( D/f_0 = 0.025 \). That is, the maximum difference between values of \( f_0 \) computed from Eq. (16) and (17) is less than the maximum observed difference in the tests and of the same order as experimental errors. It must be concluded, therefore, that the near equality of \( f_0 \) values from both conditions of equilibrium does not disprove the contention that stress-strain curves are time dependent.

Since \( f_0 \) was found from the average of values deduced from Eq. (16) and (17), Eq. (A4) and (A5) give, for strain-flexure curves,

\[
f_0 = \frac{1}{12} \left[ 7\phi(\epsilon_t) + 5 \phi'(\epsilon_t) \epsilon_t \right]
= \frac{1}{12} \left[ 7 v \gamma \left( 1 - e^{-\frac{E \epsilon_t}{v \gamma}} \right) + 5 E \epsilon_t e^{-\frac{E \epsilon_t}{v \gamma}} \right]
\]

and, again with \( v_c \gamma = 2.6 E \epsilon_t \) at \( f_0 = 0.8 f'_C \), \( f_0 = 0.77 E \epsilon_t \). On the other hand, for uniform compression at the rate \( v = v_c \), Eq. (A2) and (A7) give \( f_0 = 0.83 E \epsilon_t \). The initial tangent moduli are theoretically both equal to \( E \). Clearly, the general shape of the curves is quite similar.

The 10 percent difference found in initial moduli in the tests can best be explained on the basis of differences in the method of casting cylinders and prisms.
An apparent discrepancy with the proposed theory is that \( k_2 \) was not found exactly equal to \( 1/3 \) at \( f_c = 0 \) (Fig. 6). The only plausible explanation lies in slight differences of concrete quality in the different fibers of the prisms. At higher stresses there is, besides, a slight systematic curvature in the strain distribution. The curvature is concave downward (Fig. 9), which tends to exaggerate the computed values of \( k_2 \).

From the foregoing discussion it is justified to conclude that the actual stress distribution in the prisms was practically plane up to \( f_c = 0.8 f_c' \), and that at all stress levels it resembled the stress-strain curve of rapidly loaded cylinders rather than that of standard compression tests.

In extrapolating to reinforced concrete beams there are other factors that have been neglected in the paper. The presence of cracks in the tension zone and statistical effects play an important role which invalidates a strict extrapolation. For example, the strain distribution cannot strictly be plane, for there is a discontinuity at every crack. This would imply infinite strains in the compression zone at sections containing tension cracks. Even if the assumption of plane strain distribution is adhered to for the compression zone only, there would still be a discontinuity of shearing strains at the neutral axis, and this phenomenon was not operative in the tests on prisms.

As to statistical effects, the prisms tested tended to fail at the weakest section, whereas reinforced concrete beams will fail at the weakest section in compression, selected from only those sections which have developed tension cracks; the latter situation is undoubtedly more favorable.

Time effects, also, differ in reinforced concrete from those characteristic of the prisms. The neutral axis shifts upward and \( v \) is no longer constant for each fiber, since \( x \) decreases monotonically for all fibers as load increases.

If the foregoing objections are valid, experimental values of ultimate-strength constants should differ in the prisms and actual reinforced beams. Data in Fig. 8 show indeed such a systematic difference in experimentally determined constants. It seems likely that a trapezoidal stress distribution, such as Jensen's, with corrected \( \epsilon_0 \), would lead to simpler and more accurate prediction of the behavior of concrete in flexure than the stress-strain curve deduced from standard compression tests.

By G. M. SMITH and L. E. YOUNG*

Messrs. Hognestad, Hanson, and McHenry are to be congratulated on the ingenious and thorough method of test used in determining the shape of the stress block as associated with ultimate strength design. To really appreciate the contribution of this paper, one has only to review the literature of ultimate theories which have evolved from numerous assumed shapes of the stress block in the compression zone. Knowing the actual shape of the stress block in the compression zone will provide a more fundamental understanding of failure and possibly provide new avenues of approach to ultimate design.

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This paper and supporting evidence verify the dip in the compression stress-strain curve of concrete after ultimate stress is reached when under test in a testing machine. There is some question as to whether similar conditions exists for specimens subjected to dead loads as in a structure. The condition of dead loading of a compression member such as a cylinder would not permit a reduction in the load and therefore a reduction in stress could not occur even if the stiffness of the testing machine was infinite. In other words, do loads near failure in a testing machine simulate loads under the force of gravity? The description of test results on pp. 466 and 467 indicate that the force $P_z$ was initially increased with $P_1$ until a maximum was reached, after which $P_z$ decreased quite rapidly near failure. It would be of interest to know if the sum of $P_1$ and $P_z$ ever decreased as the specimen approached failure.

It is interesting to note that the previous wide deviation between ultimate stress in flexure and in direct compression, as described by parameter $k_3$, has been narrowed. A comparison of the stress-strain curves in flexure and direct compression in Fig. 9 and the plotted points for $k_3$ in Fig. 12 indicate that within the degree of predictability of concrete that the ultimate stresses might be identical and $k_3 = 1$.

By A. J. TAYLOR*

The authors must certainly be congratulated on a really convincing attempt to solve the problem of the compressive stress distribution at ultimate loads for members subject to flexure. But it must be realized that they have determined the solution for beams subject to some combination of axial load and moment, different from that which could actually occur in a beam, and not for pure flexural loading conditions. No doubt, in practice, few members come under the category of pure moment without axial load, but this does not appear to be ideally, the true nature of the problem attempted, which is stated in the summary at the end of the paper. It is implied that the shape of the stress block at failure for moment loads has been solved by the test methods indicated. This in fact is not the case.

ABCD (Fig. D) represents the 8 x 6 x 16-in. block under test, the loading conditions reducing to some force $P$ at an eccentricity $e$. In an attempt to represent the actual conditions of a concrete beam failing in compression, the authors have introduced a system of loading whereby it was possible to vary $P$ and $e$ such that zero strain occurred in the fibre CD, hence deducing by definition, that this was the neutral axis. The neutral axis position for an analogous concrete beam would be GH (Fig. E).

The difference between the two cases is that in the beam bending occurs about the axis GH, but in the authors’ tests bending does not occur about the axis DC, but about EF which for low loads lies at the centroid of the section. This axis EF is the “neutral axis” about which bending takes place.

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The two cases are, therefore, not analogous, since although the actual neutral axes are effectively in the same position, the bending neutral axes are in different positions but have been disguised by the axial load in the authors' tests.

The sets of curves given for $k_2$ and $k_1 k_3$ against strain (Fig. 6) are also not applicable to concrete beams subject to flexure for another reason. According to Eq. (12) and (13)

$$k_1 k_3 = \frac{P_1 + P_2}{b c f_s^c}$$

$$k_2 = 1 - \frac{P_1 a_1 + P_2 a_2}{(P_1 + P_2) c}$$

where $c$ is equal to the depth of the compressive zone in a beam but is represented by the constant total depth of the specimens tested by the authors. In an actual beam test, the depth of the compressive zone varies due to increased cracking and the introduction of plasticity. In effect, each point on the $k_2$ and $k_1 k_3$ curves represents values which could be obtained for particular loads on different beams with different effective depths. In any one of these beams the depth of the compressive zone would vary and not remain constant as in the tests.

In three out of five tests in which graphs were given for $k_2$ against concrete strain (Fig. 6), values of $k_2$ reached and exceeded a value of 1/2. With $k_2$ equal to 1/2, the line of action of the compressive force must be at the centroid of the section. For this particular condition we thus have an axially loaded specimen. It is not surprising, therefore, to find that even though moments do exist: "... there was a striking similarity between flexural stress-strain relations and those obtained in concentric compression tests on 6 x 12-in. cylinders."

AUTHORS' CLOSURE

The authors greatly appreciate the contribution made by the discussers in clarifying the fundamentals of ultimate strength design. Studies of in-
elastic concrete stress distribution have recently been intensified throughout the world, and it is gratifying that this discussion affords an opportunity for comparison of independent findings in different parts of the world.

**Comparison with other data**

Baker reports that the results of a large number of beam tests carried out at Imperial College agree closely with the results reported in the paper. By comparison with test results obtained in the USSR and in several European countries Hruban concludes that the authors' findings are applicable beyond the type of aggregate, cement, and specimen used. Though they have not had an opportunity to examine the pertinent test reports published in the USSR, the authors have recently studied the ultimate strength design specifications for reinforced concrete structures issued in that country in 1949 (N and TU-49) and in 1955 (123-55 MSPMChP).\(^1\) It was found that these specifications, which for most cases utilize an equivalent rectangular stress block, are based on fundamental concepts regarding inelastic concrete stress distribution that are strikingly similar to those presented in the paper.

While the authors' investigation was in progress, similar work was being carried out independently at the Munich Institute of Technology in West Germany under the direction of Rüsch.\(^4\) Groups of identical specimens were tested with a different and constant eccentricity for each test. Typical eccentricities were 0, 0.2, 0.6, and 1.0 times the kern distance. For all specimens within one group, strain measured at an outside face at ultimate strength was plotted versus applied eccentricity, and the eccentricity corresponding to a position of the neutral axis at an edge of the cross section was determined. By applying suitable statistical methods, information regarding the stress distribution similar to that developed by the authors was arrived at. These German findings are compared to the PCA results in Fig. F. Test results have been added for three new sand and gravel concretes and for 15 lightweight aggregate concretes tested recently by J. A. Hanson at the PCA Laboratories.

The German tests were made with sand and gravel aggregates. It is seen in Fig. F that, in spite of differences in materials and testing techniques, an excellent agreement exists between the German and the American test results for sand and gravel concretes. Furthermore, the radical change in aggregate type to lightweight materials caused only a minor change in the stress distribution properties.

**Applicability to structural concrete**

Several of the contributions to this discussion considered the extent to which the test results presented in the paper are directly and quantitatively applicable to ultimate strength design of structural concrete. The authors' position on this question was stated in the paper as "The numerical constants obtained . . . may be helpful in considering design values suitable for practical ultimate strength design," and "To make definite design recommendations, however, strong consideration must also be given to tests of reinforced concrete beams and columns . . ." Recently, the authors have had occasion
Average Stress = $k_1 k_3 f_c$

Depth to Centroid = $k_2 c$

Flexural Strain = $\varepsilon_u$

Fig. F—Ultimate strength properties of stress distribution
to compare ultimate strength computed by the numerical constants given in the paper to measured ultimate strength of a variety of structural concrete members. Agreement within a few percent has been found for columns, pre-tensioned beams, and beams of T-section, even in cases when ultimate strength was controlled by compression. Nevertheless, it is felt that the principal purpose of the test method developed should be to clarify principles and to study effects of variables in the composition of concretes, such as a change from sand and gravel to lightweight aggregates.

Smith and Young considered the effect of gravity loads. The variation of $P_1 + P_2$ during loading is reported in the paper in terms of $k_1 k_2 = (P_1 + P_2)/b c f'$. As shown in Fig. 6, a slight decrease in $k_1 k_2$ took place before final collapse of the specimen for low concrete strengths. The ultimate strength design factors given in Fig. 8 and 12 correspond to the maximum value of $P_1 + P_2$ for each test, and it is therefore believed that the same values would have been reached in a similar test under gravity loads.

Hyperstrength effects

In studies of the "hyperstrength" effects mentioned in several discussions, it is evident that consideration must be given principally to tests of reinforced members. Ashdown, Prentis, and Rosenblueth pointed out that concrete in tension between flexural cracks may contribute to ultimate beam strength. Abeles suggested a design equation accounting for such contribution, which he believes is also related to an increase in steel strength by virtue of the fact that the steel is embedded in concrete. Bjuggren pointed out the restraining effects of stirrups that may increase the strength of the compression zone in beams. He also reported observation of a similar restraining effect of concrete below the neutral axis, particularly when the distance to the neutral axis is small. Further sources of hyperstrength effects are strain hardening of the reinforcement, reentrant corners in the concrete compression zone, and catenary effects.

The authors feel that most hyperstrength effects are of an unreliable nature, and they should therefore not be considered in ultimate strength design, at least not at the present time. It is further felt that in future studies of these effects, it is advisable to carry out reinforced concrete experiments under entirely realistic conditions. It is possible, on the other hand, that development of new special test methods may be helpful in clarifying fundamental principles, for instance, regarding flexural cracking.

Mechanical models

Mechanical models such as those discussed by Cowan and Rosenblueth are valuable tools in fundamental materials research. When they are carefully developed to reflect actual properties of materials, models may aid in discovering relationships between properties of different materials, or relationships between different properties of the same material. Rosenblueth used a simple model consisting of one spring and one dashpot in series to in-
vestigate the stress-strain relationship for concrete stresses below 0.8 $f'_c$. His conclusion that the stress is less than the product of strain and initial tangent modulus is certainly correct, although the quantitative estimate may be questioned. All studies of creep in concrete with which the authors are familiar indicate that for the test conditions discussed, the creep strain would be of the order of 1 percent of the elastic strain. Prentis reported that no effect of strain rate was found in his beam tests.

The significant question in using the assumption that all "fibers" follow one and the same stress-strain curve was in the authors' view not one of time effects, but rather one of a possible effect of the stress gradient on the stress-strain relationship. As far as further developments of ultimate strength design are concerned, it is felt that the effects of creep and repeated loading may better be investigated by tests of reinforced members.

**Strains**

Rosenblueth suggests that the distribution of strain across a reinforced beam section cannot be linear at flexural cracks. This is certainly true. However, numerous tests of reinforced concrete beams and columns have shown that linear distribution of strain closely approximates reality if strains are measured over a gage length greater than the average crack spacing. The ultimate concrete strains of 0.003 to 0.005 reported in recent American literature were in most cases measured by 6-in. electric gages. Strains at a crack over a short gage length probably exceed these values. Thus, to study local conditions at cracks, the assumption of linear strain distribution is not warranted. On the other hand, this assumption is entirely reasonable in developing ultimate strength equations for reinforced concrete members.

Bjuggren raised the question regarding the extent to which creep deformations should be included in the ultimate concrete strain. For research purposes it is certainly convenient to do so. The strength of eccentrically loaded columns subject to high sustained loads has been estimated on that basis. In ultimate strength design, it is often assumed, for reinforcement with yield points not exceeding 60,000 psi, that compression reinforcement is yielding at ultimate strength. In some cases this implies a consideration of beneficial creep effects. If compression reinforcement of alloy steel quality with a yield point of about 90,000 psi is used with high-strength concrete, it seems logical to consider the beneficial effects of creep that lead to a "pre-compression" of the steel. It is felt, however, that this should be done in design only after such beneficial effects have been demonstrated by tests of reinforced members.

Abeles requested an opinion regarding the applicability of Eq. (23) to the balanced reinforcement of prestressed beams. Such application was discussed in a recent paper. To obtain an approximate value of $p_b$ by Eq. (23), $\epsilon_y$ should be taken as the yield strain minus the effective prestress strain, but $f'_y$ should be taken as the full yield point stress of the steel. The balanced percentage computed by Abeles as 2.2 percent would then be reduced to 0.8
percent, which is in reasonable agreement with the values of 0.77 to 1.08 percent he quoted from European tests. He also reported that European tests of beams with high-strength wire reinforcement have indicated higher values of \( R = \frac{M_{ult}}{bd^2f_y} \) than those corresponding to the factors given in Table 2. The authors believe this is due primarily to the low factor of 2/3 to 0.7 used by Abeles to convert the compressive strength of cubes to cylinder strengths. Had a conversion factor of 0.90 been used, the \( R \) values quoted would have been 0.245 to 0.300 which is in reasonable agreement with the value 0.270 computed by the authors’ ultimate strength factors.

On the basis that bending takes place about different axes, Taylor suggested that the authors’ test method does not represent the conditions in the compression zone of a beam. The authors feel that the important similarities between the two cases concerned are the position of the neutral axis defined as the axis of zero strain and the fact that the same gradient of stress and strain is present in both cases.

**Modulus of elasticity**

The relationship between modulus of elasticity, time, and \( c/w \) ratio suggested by Bjuggren is indeed an interesting one. It seems reasonable to expect that the constants in his equation may depend on the conditions of specimen storage.

**Straight-line theory**

Many discussions regarding adaptation of ultimate strength design concepts to everyday practice in a design office have been recorded in the literature of reinforced concrete between 1930 and the present time. If the authors interpret Hadley’s discussion correctly, he prefers to retain the straight-line method, modified in such a way as to give about the same answer with respect to measured ultimate strength as is reached by inelastic analysis. Modifications of that sort are indeed possible, as has been shown by others, but they lead to complicated design criteria which are basically irrational. Hadley’s discussion indicates that he finds it impossible to accept any evidence of “immediate plasticity” in concrete. The volume of test data involved has during the past two decades become so comprehensive and so conclusive that another paragraph or two in this closure, devoted perhaps to the “roses of yesteryear,” could scarcely serve a useful purpose.

**REFERENCES**

DISCUSSION

of the paper

CONCRETE STRESS DISTRIBUTION
IN
ULTIMATE STRENGTH DESIGN

By P. W. Abeles, A. J. Ashdown, A. L. L. Baker, Ulf Bjuggren,
Henry J. Cowan, Homer M. Hadley, Konrad Hruban, J. M.
Prentis, E. Rosenblueth, G. M. Smith, L. E. Young, A. J. Taylor,
and Authors, E. Hognestad, N. W. Hanson and D. McHenry

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Journal of the American Concrete Institute
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Portland Cement Association

D1 — “Influence of Soil Volume Change and Vegetation on Highway Engineering,” by E. J. Felt.
Reprinted from Twenty-Sixth Annual Highway Conference of the University of Colorado, May, 1963.

D2 — “Nature of Bond in Pre-Tensioned Prestressed Concrete,” by Jack R. Janney.
Reprinted from Journal of the American Concrete Institute (May, 1964); Proceedings, 50, 717 (1964).

D2A—Discussion of the paper “Nature of Bond in Pre-Tensioned Prestressed Concrete,” by P. W. Abeles, K. Hajnal-Kontti, N. W. Hanson and Author, Jack R. Janney.
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Reprinted from Journal of the American Concrete Institute (December, 1955); Proceedings, 52, 465 (1956).

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Discussion of a paper by Eivind Hognestad, N. W. Hanson, and Douglas McHenry:

Concrete Stress Distribution in Ultimate Strength Design*


By P. W. ABELES†

The extensive tests carried out by the authors represent a valuable investigation of the stress distribution in compression of concretes of different strength properties. The method of testing was ingenious and the introduction of the "critical" stress in the stress-strain diagrams for compression and flexure was interesting. This critical stress appears to be important to ensure safe values when the compressive resistance of the concrete is of direct influence upon failure, as long as the margin between actual test results and calculation is not too great. With regard to tensile failure, however, there is no need for an additional factor of safety between actual failure load and calculated load, particularly if the reinforcement is well bonded. This will result in a good crack distribution with rather fine cracks in which necking of the steel will be less than in wide cracks; consequently there will be in many cases an excess of strength as compared with the calculated value based on the yield point. This can be concluded from a great number of test results showing higher ultimate bending moments than calculated. It may be pointed out that so far nobody has been able to measure the strain of steel within a narrow crack, nor even over a correspondingly short gage length.

In view of these considerations, it seems of interest to compare the data given in the paper with actual test results. The writer has investigated this question on numerous tests, limited to high-strength materials in ordinary and prestressed concrete,¹ and an approximation, Eq. (a), has given satisfactory and safe values.

\[
R = \frac{M_{ult}}{bd''} f'_c = q (1 - 0.5 q) \quad \text{(a)}
\]

In this equation a limit of \( k_u = 0.5 \) has been taken into account for a rectangular stress block with \( k_u/k_u = 0.5 \). When comparing Eq. (a) with the authors' Eq. (3) and Table 2, we find that the ratio \( k_u/k_u \) agrees approximately only for a concrete strength of 1000 psi with the factor of 0.5, when

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it is 0.48, while with higher concrete strengths it increases up to 0.67 (for \( f'_c = 8000 \) psi), and thus the ultimate bending moment would become less than the values obtained from Eq. (a). The actual test values investigated by the writer in reference 1 were higher than those computed according to the writer's Eq. (a). Consequently he showed in reference 2 that this excess in strength, apparently due to reduced necking of the steel in the cracks, can be taken into account by considering an equivalent cooperation of the concrete in the tensile zone, based on his suggestion in the discussion to Cox's paper. 3

\[ R = \frac{M_{ut}/bd}{f'_c} = \left[ q(1 - 0.5q) - f'((t/d - 1)q - 0.5(t/d)^2) \right] / (1 + f') \]  

In Eq. (b), \( f' \) is the ratio of the equivalent concrete tensile stress, extending uniformly over the depth \((t - c)\), to the cylinder strength \( f'_c \). This equation gives higher \( R \) values in the range in which failure occurs due to yielding of the steel and thus agrees better with the test results.

For balanced design, \( q_b \) becomes 0.5 according to Eq. (a) and results in \( R_b = 0.375 \); according to Eq. (b) \( R_b \) is only slightly increased to 0.387 for the same \( q_b = 0.5 \), for a factor \( f_1 = 0.05 \) and the ratio \( t/d = 1.1 \), which results generally in

\[ R = M_{ut}/bd^2f'_c = 0.029 + 0.957q - 0.481q^2 \]  

In the following, Eq. (3), (23), and (23a) are investigated for concrete of strength \( f'_c = 6000 \) psi. According to Table 2 for this strength the following data apply: \( k_2/k_1k_3 = 0.65 \), \( k_1k_3 = 0.65 \), and \( \epsilon_s = 0.0031 \). Thus the equation equivalent to Eq. (a) becomes

\[ R = \frac{M_{ut}/bd}{f'_c} = q(1 - 0.05q) \]  

and much lower results are obtained than from Eq. (a), which has given safe values at least for high-strength materials. In the following table the authors' data given for \( f'_c = 6000 \) psi are evaluated for \( E = 30 \times 10^6 \) psi for various steel stresses, considering that high-strength steel is used and prestressing may be applied.

<table>
<thead>
<tr>
<th>( f'_c ) ksi</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
<th>100</th>
<th>120</th>
<th>150</th>
<th>200</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_2 )</td>
<td>0.700</td>
<td>0.650</td>
<td>0.608</td>
<td>0.571</td>
<td>0.537</td>
<td>0.508</td>
<td>0.482</td>
<td>0.383</td>
<td>0.319</td>
<td></td>
</tr>
<tr>
<td>( p_1 ), percent</td>
<td>6.84</td>
<td>5.07</td>
<td>3.96</td>
<td>3.07</td>
<td>2.02</td>
<td>1.92</td>
<td>1.88</td>
<td>0.99</td>
<td>0.62</td>
<td></td>
</tr>
<tr>
<td>( q_1 )</td>
<td>0.455</td>
<td>0.422</td>
<td>0.395</td>
<td>0.371</td>
<td>0.349</td>
<td>0.329</td>
<td>0.313</td>
<td>0.249</td>
<td>0.206</td>
<td></td>
</tr>
<tr>
<td>( R_b )</td>
<td>0.321</td>
<td>0.306</td>
<td>0.294</td>
<td>0.282</td>
<td>0.270</td>
<td>0.259</td>
<td>0.249</td>
<td>0.249</td>
<td>0.178</td>
<td></td>
</tr>
</tbody>
</table>

These values may be compared with some test results investigated by the writer. 1 Beam No. 24, Table IX, of the writer's tests with high-strength materials in Vienna (1935) resulted in a relatively higher value \( R = 0.246 \) for \( f'_c = 91 \) ksi and a cube strength of 8500 psi, corresponding to \( f'_c = 6000 \) psi,
the percentage being only 1.47 percent, which is much below \( p_x = 2.19\) percent computed in the above table for \( f_y = 90\) ksi.

Another comparison may be made with regard to prestressed concrete, and beams No. 10-18 of Tables I and III of reference 1 will be considered. These relate to tests carried out by Stott, Leeds, using concrete of a cube strength of 8260 psi, approximately \( f_c' = 6000\) psi, the strength of the wires being 208 ksi and the initial prestress varying between 171 and 182 ksi. The \( R\) values obtained for these tests of percentages between 0.77 and 1.08 percent varied between 0.352 and 0.425; however, a certain amount has to be deducted, since 9 to 11 tensioned wires 0.08 in diameter were placed in the compressive zone, 1/2 in. below top surface. If the ultimate bending moment is reduced by the resistance of the compressive reinforcement, computed for a steel stress of 100 ksi, the \( R\) values are reduced to 0.316-0.386. When comparing these values with those shown in the table on p. 1306, first the appropriate steel stress \( f_y\) must be selected.

There are different views about the elongation \( \varepsilon_n\) to be considered for tensioned steel in prestressed concrete. According to one school of thought, only the difference between the elongation of the steel at failure and that of the effective prestress should be taken into account. According to the other school, the entire elongation applies, whether the steel is tensioned or not. This is based on the assumption that, when failure approaches, the prestress relaxes considerably and may even become zero in a crack. If in the special case an effective prestress of 140 ksi and a yield point of 220 ksi is considered, resulting in a difference of 80 ksi, then according to the above table, \( R_x = 0.270\) and \( p_x = 2.2\) percent would apply, whereas in fact \( R\) values 0.315 to 0.386 were obtained for percentages much below 2.2 percent. This discrepancy would become even much more pronounced, if instead of the difference of elongation the entire elongation of the wire were taken into account.

It would appear that neither of the two assumptions mentioned above, related to the ultimate elongation of the steel, is correct but that an intermediate solution applies, and the writer would be glad to hear the authors' opinion on this question particularly in view of the above examples. As already mentioned, insufficient data are available about the actual elongation of the steel in a crack, which solely influences failure conditions if the beam is under-reinforced. The efficiency of the tensile resistance will, in fact, mainly depend on the bond resistance, which is less easily obtained with untensioned high-strength steel than with prestressed steel, as Janney has clearly shown.

It may be pointed out that the results obtained by the authors and summarized in Table 2 give important data about the compressive resistance of concrete. However, their application to the design of reinforced or prestressed concrete beams will have to be investigated by further tests in which the actual resistance of the tensile reinforcement is taken into account and not based on the assumed elongation at failure. These remarks refer solely to high-strength materials. For ordinary reinforced concrete with mild steel
the data given in Table 2 will most certainly give satisfactory results for ultimate load design.

REFERENCES


By A. J. ASHDOWN*

The method of determining the stress distribution in the compression zone of a concrete beam put forward by the authors represents a distinct advance in experimentation for this purpose. The results are not complicated by the interaction between concrete and steel, such as occurs in beams. The differences between these test results and those obtained from beams are quite wide as revealed by Fig. 8(a), and in my opinion this is due to insufficient appreciation of what occurs between the steel and concrete in beams, before and after cracking. The usual assumption of perfect bond does not occur; the concrete drags on the steel (especially noticeable with deformed bars), and between cracks lowers the stress in the steel, or puts it into a relative compression. Close to the support the steel may actually be in compression; if this were so, the zero for tensile stress would be lowered. This would partly account for the so-called "hyperstrength" of reinforced concrete beams. This means that, although the concrete has cracked, it is still, partly if not wholly, assisting in resisting bending through the steel between the cracks.

It may be noted, that in the authors' test piece, the compressed length is quite short compared with a beam, which would require an effective depth of at least double that of the specimen for an equivalent compressed area, so that for a third-point loaded beam, one should expect that the ultimate compressive strength values on the much longer length to be rather lower than those obtained in these tests, even taking into consideration the narrow width employed.

The wide scatter of values for the modulus of elasticity of concrete is well known. I use a simple formula for the initial modulus, \[ E_c = 52,000 \sqrt{c} \text{ or } 58,300 \sqrt[d]{f'_c} \] for cube strength or cylinder strength, respectively, and which lies between Jensen's and the proposed formula in Fig. 10.

It is gratifying to note that the strain profiles are essentially linear.

It is altogether an excellent and stimulating paper.

At Imperial College, during the last 10 years, we have carried out a large number of beam tests and obtained results which agree closely with those shown in Fig. 6. We have adopted 0.4 as a safe value of $k_2$. Using strain gages to determine the position of the neutral axis at failure, we have determined the average compressive stress in terms of the cube strength $c_s$ and found that safe values are 0.6 $c_s$ for concretes weaker than about 4000 psi and 0.5 $c_s$ for concretes stronger than about 4000 psi. Since cylinder strength is about 0.8 cube strength, these values agree well with the $k_2$ values of Fig. 6.

We have also found that 0.002 is a safe minimum value of the concrete strain at which maximum strength occurs, as shown in Fig. 2. We agree, too, with the assumption of a linear distribution of strain, except at wide cracks when the ratio of the steel strain to the virtual concrete strain equals $F$. The value of $F$ varies from about 0.8 to 1 according to bond conditions. The strain values at failure determine the position of the neutral axis, and the moment of resistance of a beam is then easily obtained from the above safe values. In bonded prestressed beams, the steel strain is assumed to be the total strain minus the strain due to prestress.

Some further tests have been carried out on the bending simulation machine referred to in the paper and agree well with the beam tests. Precise determination of the shape of the compressive stress distribution is a slow procedure, but it has been obtained. It has become, however, mainly of academic interest, since safe values of the average compressive stress and the position of the center of compression obtained from many beam tests provides sufficient information for design purposes. The position of the center of compression is determined from the lever arm value, the latter being obtained from the measured total steel tension and the applied bending moment.

This excellent paper gives rise to some reflections, which might be of general interest. The majority of reinforced concrete beams are under-reinforced and in those cases the difference in calculated ultimate bending strength according to conventional calculations and to ultimate strength design is negligible. In over-reinforced cross sections, such as columns and some types of beams with high quality steel, the compressive area takes up a rather large part of the whole cross section. Sustained loads can in that case give rise to appreciable creep in the concrete. In the usual beams the actual creep deformations will have comparatively low values, because a lowering of the neutral axis will decrease the concrete stresses to a high degree. This will not
happen to the same extent in over-reinforced cross sections. The question is: should the creep deformations be included in the ultimate concrete strain or not? If we take an extreme case of a cross section having a sustained highest compressive stress of about 2000 psi, the corresponding elastic strain is of the order 0.0004 and the creep deformation might be of the order 0.0012, which is 40 percent of the value of the ultimate strain recommended by the authors.

The average compressive stress at compression failure depends, as the authors have pointed out, quite a lot on the effect of transverse strains. Appropriately placed stirrups thus can to a certain degree increase the ultimate compressive force. A similar effect can be obtained if the compressive area of the cross section is sustained by concrete with low stresses. This can especially be the fact, when the distance to the neutral axis is small. I have found with a 12 in. high cross section (cube strength 9000 psi) that the height of the
compressive stress in the concrete was higher than the cube strength. On the other hand, the dimensions of the test specimens used by the authors correspond to the compressive area of an usual beam with a depth of about 60 in. There may be more favorable results obtained by such small beams as joists.

Finally I would like to mention the obtained values of the initial modulus of elasticity. Starting from the point of view, that in this case these values ought to depend on one quality factor and one time factor, I have arranged the values in flexure, measured from Fig. 10, as a function of the 10th logarithm of the age of the concrete and of the cement-water ratio, namely \( F(t, w/c) = (0.24 + 1.43 \log t + 0.75 c/w) \times 10^4 \), where \( t \) = age of concrete in days. As seen from Fig. A, the result seems to be significant as a first approximation.

By HENRY J. COWAN*

The paper is an outstanding contribution toward the design of reinforced and prestressed concrete for ultimate strength. In the past it has been necessary to assume a shape for the concrete stress distribution, and to prove the accuracy of the assumption by comparing the predictions of the resulting theory with the failing loads of test beams. We have now for the first time extensive experimental data of the shape of the stress-strain diagram itself.

The most significant features of results presented are the relatively high ultimate strains and the descending portion of the stress-strain diagram. Neither can be reconciled with mathematical theories of the inelastic deformation of concrete which have been published. Glanville's experiments\(^1\) in 1930 established the existence of inelastic strains which were proportional to the applied stress. As a result, a number of rheological models for the deformation of concrete were proposed\(^\dagger\) consisting of viscous dashpots and elastic springs. While these models satisfactorily reproduce the long-term deformation of concrete at moderate loads, they fail to explain the high ultimate strains and the descending portion of the stress-strain diagram.

Freudenthal\(^2\) has suggested that the departure of the concrete stress-strain diagram from visco-elastic behavior is due to breakdown of the adhesion between the cement and the aggregate. This presumes the formation of micro-cracks above a certain limit. The presence of minute cracks at about 60 percent of the ultimate load has in fact been observed by Berg,\(^3\) using an optical method, and by Jones,\(^4\) using an ultrasonic method; their presence has also been inferred from tensile tests by Blakey and Beresford.\(^5\)

These considerations lead to the statistical concept of a concrete specimen made up of a number of elements with strengths distributed over a specified range. This may be represented in a mechanical model by a number of springs rupturing progressively with increasing strain, and thereafter ceasing to contribute to the load resisting capacity of the model. Williams and Kloot, working on the behavior of timber specimens in tension,\(^6\) have pro-

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\dagger The model proposed by A. D. Ross \(\textit{Creep and Shrinkage in Plain, Reinforced, and Prestressed Concrete.}\) Journal Institution of Civil Engineers (London), V. 21, 1946, pp. 95-97 may serve as an example.
duced a mathematical theory for a model of this type. Choosing suitable parameters, a stress-strain curve obtained from this theory is shown in Fig. B. The theoretical curve obviously proceeds to an infinite strain at a very small stress, since in a statistical theory an occasional element may remain undamaged, and it is necessary in practice to stipulate a limiting ultimate strain. With this reservation, however, the agreement with the experimental data presented in this paper is evidently satisfactory, and it should be practicable to determine parameters to fit various types of experimental curves.

The mechanical model shown in Fig. C has therefore been constructed to reproduce all the known features of the inelastic deformation of concrete. It consists of two series of viscous dashpots and elastic springs in parallel, the dashpot of one of the units being blocked by nonreturn valves; and a series of brittle springs in parallel which behave elastically up to failure, the failing loads of these brittle springs being distributed over a range. When this model is deformed by a moderate load, so that the brittle springs are not damaged, it exhibits an immediate elastic deformation proportional to the applied load, and a progressive inelastic deformation at a decreasing rate, the total inelastic deformation over an infinite period being proportional to the applied load. When the model is unloaded the procedure is reversed; but due to the blocking of the second dashpot some of the inelastic deformation is not recovered. When the model is loaded sufficiently rapidly to produce negligible deformation of the dashpots, it deforms elastically until the fracturing load
of the weakest brittle spring is reached, and the subsequent inelastic deformation is irreversible.* When the model is deformed sufficiently slowly to allow for appreciable deformation of the dashpots and the load is carried at the same time above the fracturing load of the weaker brittle springs, then both types of inelastic deformation are reproduced.

It may be seen that the repeated loading of reinforced concrete structures at high loads (i.e., at loads above the working loads but below the failing loads under rapid loading) present a problem that must be investigated before ultimate strength methods can be applied to reinforced concrete structures with perfect confidence. It has been observed that inelastic deformation due to creep, by ironing out the peak moments in a redundant structure over long periods of time, increases the load bearing capacity of the concrete structure. It has similarly been observed that micro-crack formation, by extending the stress-strain diagram and increasing the ultimate strain, increases the load bearing capacity of the concrete structure loaded continuously to destruction, as compared with a structure made from a similar elastic material failing in a brittle manner at the maximum stress. If it is, however, accepted that the declining portion of the stress-strain diagram is due to irreversible micro-crack formation, then the increase in strength may not hold for repeated high loads.

The outstanding success of the authors in developing an experimental technique for determining the complete stress-strain diagram under rapid loading holds out hope for the solution of the problem of repeated loading.

REFERENCES


By HOMER M. HADLEY†

The series of tests reported in this paper is notable in its abandonment of the standard 6 x 12-in. concrete cylinder as the means of demonstrating the immediate plasticity of concrete in response to overload. Instead of the

*In practice some of the micro-cracks would probably close up due to autogenous healing, if sufficient time is allowed.
†Hadley & Hadley, Consulting Engineers, Seattle, Wash.
cylinder there is substituted the channel-shaped frame, simultaneously subjected to direct stress and bending, and a new foundation for ultimate strength design is presented.

The results of tests of these frames and of companion cylinders is presented in Fig. 9. There is a marked similarity in the stress-strain curves of two types of specimen. Perhaps the most significant feature of the curves is their limited extent; except for those of lowest strength, none reach far beyond strains of 0.003 in. per in. "Where is the rose of yesteryear?": those strains of 0.009 and 0.010 in. per in. that once graced and adorned the outer reaches of "the plastic range"? As a matter of fact, between the 0.002 strains of the elastic range and the 0.003 to 0.0035 boundaries of the elastic range, not much space is left for plasticity.

It is noted that about 15 min was given to the testing of each frame. This permits a more gradual application of load and more gradual adjustment to load on the part of frame than would be possible in a briefer time. The writer is of the opinion that the sudden application of a small increment of load to any of these frames that had attained a strain of 0.0025 in. per in. would have abruptly terminated both the curve and the frame at that point. The weakness and "come-apartness" of concrete so strained in a brief period of time has not been lost by changing the form of the specimen. The identity of those frames shown in Fig. 5 is not disclosed but the shattered condition they are in was no sudden development preceded by a relatively lengthy period of plastic integrity.

Not the least unsatisfactory feature of this ultimate strength theory is the fact that there is no definite and decisive physical boundary to limit its supposed plastic range. The elastic theory has the maximum cylinder strength which is something positive and specific and determinable. This theory has neither a definite strain, nor a definite state of cracking and shattering, nor a definite strength after a definite time of applying load at a definite rate. Its physical basis is vague and uncertain and indefinite. We have been offered the Saliger speculation, the curves extending to strains of 0.009 and 0.010 without word of the concrete's coming apart, and others. It is to be hoped that the authors will state their recommendations respecting the outer limits of the plastic range.

Fig. 2, titled "Tests of 3 x 6-in. concrete cylinders—U. S. Bureau of Reclamation," has a familiar look. On comparison the curves are found to be identical with those of Fig. 3, ACI Journal, Oct. 1952, Proc. V. 49, p. 122, where they appear in Mr. Hoghestad's paper, "Inelastic Behavior in Tests of Eccentrically Loaded Short Reinforced Concrete Columns." The chief difference between those two figures is that the curves are now designated, top to bottom: "41 days, 13 days, 9 days." These designations now definitely identify the curves with those bearing the same descriptions in Fig. 8 of Laboratory Report No. SP-12, Mar. 7, 1947, U. S. Bureau of Reclamation, wherein we learn that "each curve represents the loading of a single 3 x 6-in. cylinder." These then are the cylinders whose stress-strain relationships
for concrete in compression Professor Hognestad believed applicable to flexure and which he then took as the basis of "the present analysis," i.e., that of 1952.

Under the circumstances prevailing it is scarcely to be wondered at that L. P. Brice's experimental efforts\(^7\) to determine flexural stress-distribution in a beam receive the somewhat hurried brush-off that they do in this paper "... a straight-line stress-distribution was found, even at high loads. The principal weakness of this experimental method is the unknown extent to which the introduction of the steel plates locally changes the stress conditions in the concrete."

It is far from clear to the writer that the introduction of steel plates in Brice's beam did actually change the stress condition in the concrete appreciably. But if it did, why of all things did the stress pattern so perversely revert to one with straight-line variation, after all the labored efforts that have been made to have it a "plastic" pattern? In any event it is probably well that this beam be described in some detail so that "its principal weakness" can be more widely evaluated.

Brice's test beam is a novel and most interesting one. Its dimensions are slight: 7 x 16 x 150 cm. It was tested on a 130-cm span with equal, approximately third-point loads. At midspan a stack of eight steel plates, 14 cm long, replaced that length of concrete in the upper two-thirds of the section and an open gap or hole, about 8 cm long, was created in the concrete beneath them, across which opening the reinforcing bar extended. The eight plates, 1 cm thick, were not in contact with one another but were separated by spaces of 0.413 cm. These plates were placed and the concrete was cast against their ends and against the little forms which excluded it from the central open gap portion beneath them. In plan view these plates were hourglass shaped, of full beam width, 7 cm, at their ends narrowing to a section 1 cm wide, at the middle, which reduced section extended for a length of 45 cm. Each plate had two small projecting prongs at each end for anchoring it into the concrete and holding it in position.

Thus, to repeat, there is created a concrete beam with reinforcement continuous from end to end, with a midspan zone of constant moment and zero shear. At midspan there is a short length from which concrete is completely excluded, the gap being crossed by the reinforcing bar at the bottom and at the top by a set of steel plates about 5/8 in. thick having about 5/8 in. gap between them. The ends of these plates bear against the full width of the concrete but each plate reduces, as described, to a central section 5/8 in. square by 1 3/4 in. long, completely out of contact with its neighbors.

Such a beam appears innocent enough. It is only the concrete, pressing against the steel plates on their end surfaces that produces the recorded deformations in the steel. The steel, of course, reacts against the concrete but it is the pressure, the force, exerted by the concrete, not the strains and deformations of the concrete, that produce the deformations in the steel and these steel deformations are not so large as to involve any question as to their in-
terprctation: they are purely elastic. Now it is true that the plates show a
straight-line stress distribution pattern “even at high loads,” but even so
that fact is not, *per se*, discreditable to them, does not convincingly demon-
strate that they have “locally changed stress conditions in the concrete.”
All they have done is conflict flagrantly with the theory which our authors
seek to establish. Another possible view is that Brice has developed a simple,
direct, and most ingenious method of determining what the stresses in con-
crete actually are, quite independent of concrete deformations and the inter-
pretation thereof. The installation of a set of Brice plates in the midsection
of a concrete channel frame might present its difficulties but with the ordinary
beam there is little or no trouble to be foreseen. It is to be hoped, therefore,
that this promising field of investigation will be thoroughly and open-mindedly
explored, both by the authors and by others.

Some 25 years ago Slater and Lyse found the flexural strength of concrete,
based on the straight-line theory, in sufficiently reinforced concrete beams
to be 67 percent higher than that of 6 x 12-in. cylinders with 2000-psi concrete;
50 percent higher with 3000-psi and stronger concrete. Nearly 30 years ago
Gonnerman’s tests showed that the strength of concrete in 6 in. diameter
cylinders varied inversely with the height of the cylinder, in a fashion not
dissimilar to the tones of a trombone under varying degrees of extension. Just
why the compressive strength of the 12 in. high cylinder should be expected
to be the exact or even approximate measure of the flexural strength of its
concrete *rather* than that the strength of a 6 in. high cylinder should, *say*, or the
strength of some other height of cylinder, cannot rationally be explained but
the widespread prevalence of that belief is not to be questioned. Consequently, if the
maximum assumed value of stress in flexure is to be no more than that of the
cylinder, the only way the beam’s flexural capacity can be explained is by
scraping the straight-line theory and providing a substitute theory that
activates the region in the vicinity of the neutral axis which the straight-line
theory excludes from consideration. On the other hand, if one can entertain
the thought of a flexural unit stress greater than the compressive unit strength of
the 12 in. high cylinder by a Slater-Lyse amount, then the straight-line theory
works admirably and agrees with the beam performance. The straight-line
theory being the one with which everyone is familiar, it seems much simpler
and better to change the value of $f_c$ than to scrap the straight-line theory
altogether and to adopt a new one founded on the wonder tale of concrete’s
instant plasticity. Therefore despite the addition of a new section on ul-
imate strength design in the ACI Building Code and other “party line” efforts,
it may be advisable to hold back a bit pending a proper investigation rather
than an arbitrary rejection of Brice’s findings, which, as stated, showed a
straight-line variation of stress “even at high loads.”

Would it not, at this stage of things, be a most amazing development if
additional Brice-beams should reveal naught but straight-line stress variation?
The investigation by the authors forms a conclusive contribution to the ultimate strength theory, which has been achieved by means of an original and ingenious method. I would like to make a few comments to the authors’ remark on p. 477 that “the tests cover only one type of aggregate, one maximum aggregate size, and one size of specimen without compression reinforcement.”

I compared the test data given in the paper with results obtained elsewhere. In the USSR, for instance, eccentrically loaded columns of rectangular and I-shape sections have been tested up to considerable dimensions; the largest specimen had a section of 80 x 160 cm (about 32 x 64 in.). Comparative tests were carried out on columns, the sections of which had areas twice to 64 times smaller.¹ Other investigations have been carried out in several European countries, different types of aggregates and cements have been used, and different testing arrangements applied.

All the accumulated evidence leads incontestably to the conclusion that the stress-strain relationship for concrete in flexure is nonlinear and has to be represented by curves similar to those shown in Fig. 9. I think, therefore, the authors’ investigation gives qualitative information of fairly general validity.

Let us now compare the quantitative test data arrived at in the paper. Eq. (3) may be rewritten in the form

\[ M_{ult} = A_t f_t d \left( 1 - \frac{k_2}{k_1} \frac{f_t}{f_t'} \right) \]  

The value of \( k_2/k_1 \) depends on the concrete strength; according to Table 2 it varied between 0.48 and 0.67. The difference is significant. In Europe, we consider it in practical computations in the following way. We substitute a rectangular stress block for the actual one. To obtain the correct moment \( M_{ult} \) given by Eq. (a), both the actual and the substituting stress block must have the same volume and give the same line of action of the total compression \( C \) (see Fig. 1). This means for a rectangular section that the substituting rectangular stress area has a depth of \( 2k_2 f_t \). Denoting with \( f_t' \) the assumed uniform stress in the pressure zone (the so-called flexural strength of concrete) we obtain, instead of Eq. (a), the expression

\[ M_{ult} = A_t f_t d \left( 1 - \frac{1}{2} \frac{f_t}{f_t'} \right) \]

Both Eq. (a) and (b) yield the same result, if

\[ \frac{k_2}{k_1} = \frac{1}{2} \frac{f_t'}{f_t} \]

The standard specifications of these European countries, which have adopted the ultimate strength design, determine simply the value of \( f_t' \) for

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¹ Other investigations have been carried out in several European countries, different types of aggregates and cements have been used, and different testing arrangements applied.
each category of concrete so that it corresponds to the test value of \( k_2/k_3 \) according to Eq. (c). The category of concrete is defined by the 20-cm cube strength of concrete.

In this country, for instance, the standard values of \( f'_{lmax} \) (given in the 1955 specifications\(^1\)) have been determined according to the following formula, which was suggested by the discusser with regard to previous test evidence:

\[
f'_{lmax} = \frac{1455 f'_{\text{cube}}}{1490 + f'_{\text{cube}}} \tag{d}
\]

Here, \( f'_{\text{cube}} \) is the 20-cm cube strength of concrete in kg per sq cm. To compare the test results, from which Eq. (d) has been derived, with those given by the authors, we have to take into account that 1 psi = 0.0703 kg per sq cm and that \( f'_{\text{cube}} \) is regarded as being equal to 1.05 \( f' \).

Using these relations and substituting into Eq. (c), we obtain the values of \( k_2/k_3 \) corresponding to the Middle European tests. The results are given in the third column of Table A.

Evidently, both groups of tests lead to practically the same conclusions; the greatest difference between both respective numbers is 3 to 4 percent of the authors' values.

Russian standard specifications\(^3\) of 1948 give values of the strength \( f'_{lmax} \) which are 5 to 7 percent higher than those calculated by means of Eq. (d). This has but an insignificant influence on the resulting ultimate moment given by Eq. (b); the difference against Eq. (a) does not reach 3 percent.

It can therefore be stated that the findings of the authors are also quantitatively in accord with European investigations. The small variations can easily be explained by local influences:

In my opinion the conclusions arrived at in the paper may be considered as a definitive answer to the question of the concrete stress distribution.

REFERENCES

3. GOST 4286-48 (USSR standard specifications) and NITU-3-48 (Russian norms and regulations), Moscow, 1948 (in Russian).

By J. M. PRENTIS*

On p. 462 the authors point out that their method has advantages over the methods of analyzing beam test data proposed by Prentis, Hamman, and Lee for it avoids the complications introduced by the presence of the rein-

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forcement and the concrete tensile stresses. There is a further advantage which, in some respects, is more important than the two already mentioned, for the difficulty of measuring the steel strain can be overcome* and the tensile stresses are generally insignificant in their effect. There are, however, inherent difficulties in measuring accurately the concrete strains in a beam, due to the finite and often irregular spacing and form of the tensile cracks. It is necessary to measure average strain readings over a 6- or 8-in. gage length to determine the maximum concrete strain ($\varepsilon_c$) and the position of the neutral axis. Thus, the section of failure $e_c$ is somewhat greater than the mean value and the depth of the compression zone is less than that indicated by the strain gages. Furthermore, it is difficult to estimate the extent to which finite spacing of cracks upset distribution of stress above neutral axis.

In deliberately ensuring that no tensile stresses are generated the authors avoid these difficulties. The writer would like to congratulate the authors on the ingenious way in which they have done this.

However, having isolated the problem of determining the concrete stress from the complications due to reinforcement, concrete tensile stresses, and crack spacing there appears to be a logical difficulty in applying the results to normal reinforced concrete beams where all these factors are present. The writer would like to suggest that it is desirable that further tests be carried out in which tensile cracks are allowed to form.

A comparison of the results from tests on pairs of identical specimens in which one of a pair is tested without cracks while its mate is tested with cracks would demonstrate the effect to which the crack spacing and presence of tensile stresses upsets the analysis. The writer must confess to a personal interest here, such tests would indicate the extent to which the beam analyses of Prentis, Flannan, and Lee are well founded.

One of the basic assumptions of all the methods is that concrete stress is a function of strain only, time and stress history effects being negligible. The validity of this is questionable but it appears that the authors could use their test methods to verify it. It would be possible to test identical specimens at different rates of loading; or alternatively, single specimens could be tested so that the strain at one edge was held at, say, twice the strain at the other and so strained twice as fast. If time effects are negligible then the resultant stress-strain curves from such tests would be the same whatever the rate of straining.

Beam tests by the writer have provided evidence substantiating this assumption. These tests are referred to in a paper by the writer, but the relevant points may be mentioned here. A number of beams were tested under four-point loading to give a central span subjected to a uniform bending moment. With each beam it was possible to measure strains at three sections along the length of the central span. Due to irregular cracking the compressive strain in the concrete and the position of the neutral axis varied between

sections. Using the writer's method of analyzing the data the stress distribution was drawn for each of the three sections. Although the strain distribution, and hence the rate of straining, differed for each section the stress-strain curves were found to be the same. The time taken for testing each beam was 4 to 5 hr. Had time effects not been negligible this would have resulted in a different stress-strain curve for each section.

By E. ROSENBLUETH*

The paper constitutes an important contribution to the theory of ultimate strength design. This discussion should not detract from the main conclusion of the paper, namely, that the proposed curved distribution of stresses in concrete leads to fairly accurate prediction of flexural capacity in prismatic members of rectangular cross section. Nevertheless, it does warn against unwarranted extrapolation and neglect of time effects.

Conclusions derived in the paper may be grouped in two categories. The first concerns stress distribution in prisms loaded similarly to those reported in the paper. The methods used for computing stress-strain curves in the prisms are objectionable principally because of the assumption that concrete stress is a function of strain only and does not depend on the history or rate of loading. It is well established that stress-strain relations in concrete are strongly time sensitive. Now, the strain rate in the prisms varies approximately in proportion to the distance from the neutral axis, where it is zero. Admitting a linear variation of \( \varepsilon \), one may write

\[
\dot{\varepsilon} = \frac{\varepsilon}{c} \text{......................................................... (A1)}
\]

where the dot denotes derivative with respect to time. The effects of such a pronounced variation in strain rate should certainly not be neglected.

To gain some idea as to effects of the variable strain rate, it will be assumed that concrete may be represented as a certain combination of Maxwell and Kelvin bodies. That is, the behavior of each element of concrete will be idealized as that of a system of linear springs and viscous dampers. Under these conditions, extrapolation to \( \varepsilon = \infty \) from direct compression tests performed at different rates should give a linear relation between stress and strain. Glanville,† as well as several other investigators, has shown conclusively that this is nearly the case, at least up to stresses equal to 0.8 \( f' \). Admitting the validity of the proposed idealization in the range 0 \( \leq f \leq 0.8 f' \), the stress-strain relation can be put in the form

\[
f = \phi (\varepsilon) \text{......................................................... (A2)}
\]

where \( \phi \) depends on the strain rate and strain history.

Again, assuming linear strain distribution in the prisms, it is simple to show that

\[
\dot{\varepsilon} = \frac{d (\varepsilon)}{c} \text{......................................................... (A3)}
\]

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and the stress distribution is linear at all times up to $f_c = 0.8 f'$. From Eq. (A3)

$$f_c = \frac{1}{2} \int_0^\infty f_c \, dx = \frac{1}{2} \phi(\varepsilon_c)$$

and from Eq. (16)

$$f_c = \frac{1}{2} \left[ \phi(\varepsilon_c) + \phi'(\varepsilon_c) \varepsilon_c \right]$$

where the prime denotes derivative with respect to $\varepsilon$. Similarly, from Eq. (17)

$$f_c = \frac{1}{3} \left[ 2\phi(\varepsilon_c) + \phi'(\varepsilon_c) \varepsilon_c \right]$$

The difference between values of $f_c$ computed from Eq. (A4) and (A5) is

$$D = \frac{1}{6} \left[ \phi(\varepsilon_c) - \phi'(\varepsilon_c) \varepsilon_c \right]$$

It is true that $D = 0$ only when $\phi'/\varepsilon_c$ is constant, which corresponds to no internal damping. However, $D/f_c$ is quite small in practice. For example, in the range covered by the tests one may take, in first approximation, each concrete element idealized as a simple spring and dashpot in series. Then, for a constant strain rate $\dot{\varepsilon} = \nu$

$$\phi = \gamma \left( 1 - e^{-\frac{E}{v\gamma}} \right)$$

where $\gamma$ and $E$ represent the damper and spring constants, respectively. Substituting in Eq. (A4) to (A6) and assuming $v\gamma = 2.6 E\varepsilon_c$ at $f_c = 0.8 f'$, which is representative of the curves in Fig. 9, one obtains $D/f_c = 0.025$. That is, the maximum difference between values of $f_c$ computed from Eq. (16) and (17) is less than the maximum observed difference in the tests and of the same order as experimental errors. It must be concluded, therefore, that the near equality of $f_c$ values from both conditions of equilibrium does not disprove the contention that stress-strain curves are time dependent.

Since $f_c$ was found from the average of values deduced from Eq. (16) and (17), Eq. (A4) and (A5) give, for strain-flexure curves,

$$f_c = \frac{1}{12} \left[ 7\phi(\varepsilon_c) + 5 \phi'(\varepsilon_c) \varepsilon_c \right]$$

and, again with $v\gamma = 2.6 E\varepsilon_c$ at $f_c = 0.8 f'$, $f_c = 0.77 E\varepsilon_c$. On the other hand, for uniform compression at the rate $\nu = v\varepsilon_c$, Eq. (A2) and (A7) give $f_c = 0.83 E\varepsilon_c$. The initial tangent moduli are theoretically both equal to $E$. Clearly, the general shape of the curves is quite similar.

The 10 percent difference found in initial moduli in the tests can best be explained on the basis of differences in the method of casting cylinders and prisms.
An apparent discrepancy with the proposed theory is that \( k_a \) was not found exactly equal to 1/3 at \( f_c = 0 \) (Fig. 6). The only plausible explanation lies in slight differences of concrete quality in the different fibers of the prisms. At higher stresses there is, besides, a slight systematic curvature in the strain distribution. The curvature is concave downward (Fig. 9), which tends to exaggerate the computed values of \( k_a \).

From the foregoing discussion it is justified to conclude that the actual stress distribution in the prisms was practically plane up to \( f_c = 0.8 f_c' \), and that at all stress levels it resembled the stress-strain curve of rapidly loaded cylinders rather than that of standard compression tests.

In extrapolating to reinforced concrete beams there are other factors that have been neglected in the paper. The presence of cracks in the tension zone and statistical effects play an important role which invalidates a strict extrapolation. For example, the strain distribution cannot strictly be plane, for there is a discontinuity at every crack. This would imply infinite strains in the compression zone at sections containing tension cracks. Even if the assumption of plane strain distribution is adhered to for the compression zone only, there would still be a discontinuity of shearing strains at the neutral axis, and this phenomenon was not operative in the tests on prisms.

As to statistical effects, the prisms tested tended to fail at the weakest section, whereas reinforced concrete beams will fail at the weakest section in compression, selected from only those sections which have developed tension cracks; the latter situation is undoubtedly more favorable.

Time effects, also, differ in reinforced concrete from those characteristic of the prisms. The neutral axis shifts upward and \( v \) is no longer constant for each fiber, since \( x \) decreases monotonically for all fibers as load increases.

If the foregoing objections are valid, experimental values of ultimate-strength constants should differ in the prisms and actual reinforced beams. Data in Fig. 8 show indeed such a systematic difference in experimentally determined constants. It seems likely that a trapezoidal stress distribution, such as Jensen's, with corrected \( e_a \) would lead to simpler and more accurate prediction of the behavior of concrete in flexure than the stress-strain curve deduced from standard compression tests.

By G. M. SMITH and L. E. YOUNG*

Messrs. Hognestad, Hanson, and McHenry are to be congratulated on the ingenious and thorough method of test used in determining the shape of the stress block as associated with ultimate strength design. To really appreciate the contribution of this paper, one has only to review the literature of ultimate theories which have evolved from numerous assumed shapes of the stress block in the compression zone. Knowing the actual shape of the stress block in the compression zone will provide a more fundamental understanding of failure and possibly provide new avenues of approach to ultimate design.

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This paper and supporting evidence verify the dip in the compression stress-strain curve of concrete after ultimate stress is reached when under test in a testing machine. There is some question as to whether similar conditions exists for specimens subjected to dead loads as in a structure. The condition of dead loading of a compression member such as a cylinder would not permit a reduction in the load and therefore a reduction in stress could not occur even if the stiffness of the testing machine was infinite. In other words, do loads near failure in a testing machine simulate loads under the force of gravity? The description of test results on pp. 466 and 467 indicate that the force $P_z$ was initially increased with $P_1$ until a maximum was reached, after which $P_z$ decreased quite rapidly near failure. It would be of interest to know if the sum of $P_1$ and $P_z$ ever decreased as the specimen approached failure.

It is interesting to note that the previous wide deviation between ultimate stress in flexure and in direct compression, as described by parameter $k_3$, has been narrowed. A comparison of the stress-strain curves in flexure and direct compression in Fig. 9 and the plotted points for $k_3$ in Fig. 12 indicate that within the degree of predictability of concrete that the ultimate stresses might be identical and $k_3 = 1$.

By A. J. TAYLOR*

The authors must certainly be congratulated on a really convincing attempt to solve the problem of the compressive stress distribution at ultimate loads for members subject to flexure. But it must be realized that they have determined the solution for beams subject to some combination of axial load and moment, different from that which could actually occur in a beam, and not for pure flexural loading conditions. No doubt, in practice, few members come under the category of pure moment without axial load, but this does not appear to be ideally, the true nature of the problem attempted, which is stated in the summary at the end of the paper. It is implied that the shape of the stress block at failure for moment loads has been solved by the test methods indicated. This in fact is not the case.

ABCD (Fig. D) represents the 8 x 6 x 16-in. block under test, the loading conditions reducing to some force $P$ at an eccentricity $e$. In an attempt to represent the actual conditions of a concrete beam failing in compression, the authors have introduced a system of loading whereby it was possible to vary $P$ and $e$ such that zero strain occurred in the fibre CD, hence deducing by definition, that this was the neutral axis. The neutral axis position for an analogous concrete beam would be GH (Fig. E).

The difference between the two cases is that in the beam bending occurs about the axis GH, but in the authors' tests bending does not occur about the axis DC, but about EF which for low loads lies at the centroid of the section. This axis EF is the "neutral axis" about which bending takes place.

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The two cases are, therefore, not analogous, since although the actual neutral axes are effectively in the same position, the bending neutral axes are in different positions but have been disguised by the axial load in the authors' tests.

The sets of curves given for \( k_2 \) and \( k_1 k_3 \) against strain (Fig. 6) are also not applicable to concrete beams subject to flexure for another reason. According to Eq. (12) and (13)

\[
\begin{align*}
    k_1 k_3 &= \frac{P_1 + P_2}{bcf_c'} \\
    k_2 &= 1 - \frac{P_1 a_1 + P_2 a_2}{(P_1 + P_2) c}
\end{align*}
\]

where \( c \) is equal to the depth of the compressive zone in a beam but is represented by the constant total depth of the specimens tested by the authors. In an actual beam test, the depth of the compressive zone varies due to increased cracking and the introduction of plasticity. In effect, each point on the \( k_2 \) and \( k_1 k_3 \) curves represents values which could be obtained for particular loads on different beams with different effective depths. In any one of these beams the depth of the compressive zone would vary and not remain constant as in the tests.

In three out of five tests in which graphs were given for \( k_2 \) against concrete strain (Fig. 6), values of \( k_2 \) reached and exceeded a value of 1/2. With \( k_2 \) equal to 1/2, the line of action of the compressive force must be at the centroid of the section. For this particular condition we thus have an axially loaded specimen. It is not surprising, therefore, to find that even though moments do exist: "... there was a striking similarity between flexural stress-strain relations and those obtained in concentric compression tests on 6 x 12-in. cylinders."

**AUTHORS' CLOSURE**

The authors greatly appreciate the contribution made by the discussers in clarifying the fundamentals of ultimate strength design. Studies of
elastic concrete stress distribution have recently been intensified throughout
the world, and it is gratifying that this discussion affords an opportunity for
comparison of independent findings in different parts of the world.

Comparison with other data

Baker reports that the results of a large number of beam tests carried out
at Imperial College agree closely with the results reported in the paper. By
comparison with test results obtained in the USSR and in several European
countries Hruban concludes that the authors' findings are applicable beyond
the type of aggregate, cement, and specimen used. Though they have not
had an opportunity to examine the pertinent test reports published in the
USSR, the authors have recently studied the ultimate strength design specifi-
cations for reinforced concrete structures issued in that country in 1949 (N
and TU-49) and in 1955 (123-55 MSPMChP). It was found that these
specifications, which for most cases utilize an equivalent rectangular stress
block, are based on fundamental concepts regarding inelastic concrete stress
distribution that are strikingly similar to those presented in the paper.

While the authors' investigation was in progress, similar work was being
carried out independently at the Munich Institute of Technology in West
Germany under the direction of Rüsch. Groups of identical specimens
were tested with a different and constant eccentricity for each test. Typical
eccentricities were 0, 0.2, 0.6, and 1.0 times the kern distance. For all speci-
mens within one group, strain measured at an outside face at ultimate strength
was plotted versus applied eccentricity, and the eccentricity corresponding
to a position of the neutral axis at an edge of the cross section was determined.
By applying suitable statistical methods, information regarding the stress
distribution similar to that developed by the authors was arrived at. These
German findings are compared to the PCA results in Fig. F. Test results
have been added for three sand and gravel concretes and for 15 lightweight
aggregate concretes tested recently by J. A. Hanson at the PCA Laboratories.
The German tests were made with sand and gravel aggregates. It is seen
in Fig. F that, in spite of differences in materials and testing techniques, an
excellent agreement exists between the German and the American test results
for sand and gravel concretes. Furthermore, the radical change in aggregate
type to lightweight materials caused only a minor change in the stress distri-
bution properties.

Applicability to structural concrete

Several of the contributions to this discussion considered the extent to
which the test results presented in the paper are directly and quantitatively
applicable to ultimate strength design of structural concrete. The authors'
position on this question was stated in the paper as "The numerical con-
stants obtained... may be helpful in considering design values suitable for
practical ultimate strength design," and "To make definite design recommenda-
tions, however, strong consideration must also be given to tests of reinforced
concrete beams and columns..." Recently, the authors have had occasion
Average Stress = $k_1 k_3 f'_c$

Depth to Centroid = $k_2 c$

Flexural Strain = $\varepsilon_u$

Fig. F—Ultimate strength properties of stress distribution
to compare ultimate strength computed by the numerical constants given in
the paper to measured ultimate strength of a variety of structural concrete
members. Agreement within a few percent has been found for columns,
pre-tensioned beams, and beams of T-section, even in cases when ultimate
strength was controlled by compression. Nevertheless, it is felt that the
principal purpose of the test method developed should be to clarify principles
and to study effects of variables in the composition of concretes, such as a
change from sand and gravel to lightweight aggregates.

Smith and Young considered the effect of gravity loads. The variation of
$P_1 + P_2$ during loading is reported in the paper in terms of $k_1 k_3 = (P_1 + P_2) / \beta c f'_c$. As shown in Fig. 6, a slight decrease in $k_1 k_3$ took place before final
collapse of the specimen for low concrete strengths. The ultimate strength
design factors given in Fig. 8 and 12 correspond to the maximum value of
$P_1 + P_2$ for each test, and it is therefore believed that the same values would
have been reached in a similar test under gravity loads.

Hyperstrength effects

In studies of the “hyperstrength” effects mentioned in several discussions,
it is evident that consideration must be given principally to tests of rein-
forced members. Ashdown, Prentis, and Rosenblueth pointed out that con-
crete in tension between flexural cracks may contribute to ultimate beam
strength. Abeles suggested a design equation accounting for such contribu-
tion, which he believes is also related to an increase in steel strength by virtue
of the fact that the steel is embedded in concrete. Bjuggren pointed out the
restraining effects of stirrups that may increase the strength of the com-
pression zone in beams. He also reported observation of a similar restraining
effect of concrete below the neutral axis, particularly when the distance to the
neutral axis is small. Further sources of hyperstrength effects are strain
hardening of the reinforcement, reentrant corners in the concrete compression
zone, and catenary effects.

The authors feel that most hyperstrength effects are of an unreliable nature,
and they should therefore not be considered in ultimate strength design, at
least not at the present time. It is further felt that in future studies of these
effects, it is advisable to carry out reinforced concrete experiments under en-
tirely realistic conditions. It is possible, on the other hand, that development
of new special test methods may be helpful in clarifying fundamental prin-
ciples, for instance, regarding flexural cracking.

Mechanical models

Mechanical models such as those discussed by Cowan and Rosenblueth
are valuable tools in fundamental materials research. When they are care-
fully developed to reflect actual properties of materials, models may aid in
discovering relationships between properties of different materials, or rela-
tionships between different properties of the same material. Rosenblueth
used a simple model consisting of one spring and one dashpot in series to in-
investigate the stress-strain relationship for concrete stresses below 0.8 \( f' \). His conclusion that the stress is less than the product of strain and initial tangent modulus is certainly correct, although the quantitative estimate may be questioned. All studies of creep in concrete with which the authors are familiar indicate that for the test conditions discussed, the creep strain would be of the order of 1 percent of the elastic strain. Prentis reported that no effect of strain rate was found in his beam tests.

The significant question in using the assumption that all “fibers” follow one and the same stress-strain curve was in the authors’ view not one of time effects, but rather one of a possible effect of the stress gradient on the stress-strain relationship. As far as further developments of ultimate strength design are concerned, it is felt that the effects of creep and repeated loading may better be investigated by tests of reinforced members.

**Strains**

Rosenblueth suggests that the distribution of strain across a reinforced beam section cannot be linear at flexural cracks. This is certainly true. However, numerous tests of reinforced concrete beams and columns have shown that linear distribution of strain closely approximates reality if strains are measured over a gage length greater than the average crack spacing. The ultimate concrete strains of 0.003 to 0.005 reported in recent American literature were in most cases measured by 6-in. electric gages. Strains at a crack over a short gage length probably exceed these values. Thus, to study local conditions at cracks, the assumption of linear strain distribution is not warranted. On the other hand, this assumption is entirely reasonable in developing ultimate strength equations for reinforced concrete members.

Bjuggren raised the question regarding the extent to which creep deformations should be included in the ultimate concrete strain. For research purposes it is certainly convenient to do so. The strength of eccentrically loaded columns subject to high sustained loads has been estimated on that basis. In ultimate strength design, it is often assumed, for reinforcement with yield points not exceeding 60,000 psi, that compression reinforcement is yielding at ultimate strength. In some cases this implies a consideration of beneficial creep effects. If compression reinforcement of alloy steel quality with a yield point of about 90,000 psi is used with high-strength concrete, it seems logical to consider the beneficial effects of creep that lead to a “pre-compression” of the steel. It is felt, however, that this should be done in design only after such beneficial effects have been demonstrated by tests of reinforced members.

Abeles requested an opinion regarding the applicability of Eq. (23) to the balanced reinforcement of prestressed beams. Such application was discussed in a recent paper. To obtain an approximate value of \( \rho_b \) by Eq. (23), \( \epsilon_y \) should be taken as the yield strain minus the effective prestress strain, but \( f_y \) should be taken as the full yield point stress of the steel. The balanced percentage computed by Abeles as 2.2 percent would then be reduced to 0.8
percent, which is in reasonable agreement with the values of 0.77 to 1.08 percent he quoted from European tests. He also reported that European tests of beams with high-strength wire reinforcement have indicated higher values of $R = \frac{M_{ult}}{bd^2f'}$ than those corresponding to the factors given in Table 2. The authors believe this is due primarily to the low factor of $2/3$ used by Abeles to convert the compressive strength of cubes to cylinder strengths. Had a conversion factor of 0.90 been used, the $R$ values quoted would have been 0.245 to 0.300 which is in reasonable agreement with the value 0.270 computed by the authors' ultimate strength factors.

On the basis that bending takes place about different axes, Taylor suggested that the authors' test method does not represent the conditions in the compression zone of a beam. The authors feel that the important similarities between the two cases concerned are the position of the neutral axis defined as the axis of zero strain and the fact that the same gradient of stress and strain is present in both cases.

**Modulus of elasticity**

The relationship between modulus of elasticity, time, and c/w ratio suggested by Bjuggren is indeed an interesting one. It seems reasonable to expect that the constants in his equation may depend on the conditions of specimen storage.

**Straight-line theory**

Many discussions regarding adaptation of ultimate strength design concepts to everyday practice in a design office have been recorded in the literature of reinforced concrete between 1930 and the present time. If the authors interpret Hadley's discussion correctly, he prefers to retain the straight-line method, modified in such a way as to give about the same answer with respect to measured ultimate strength as is reached by inelastic analysis. Modifications of that sort are indeed possible, as has been shown by others, but they lead to complicated design criteria which are basically irrational. Hadley's discussion indicates that he finds it impossible to accept any evidence of "immediate plasticity" in concrete. The volume of test data involved has during the past two decades become so comprehensive and so conclusive that another paragraph or two in this closure, devoted perhaps to the "roses of yesteryear," could scarcely serve a useful purpose.

**REFERENCES**